ETH, D-MATH

4-Manifolds

and their Intersection Forms

Niclas Kupper

Supervisor

Will J. Merry

August 7, 2018

Contents

0 Introduction

Consider the sphere and the torus. One way to tell them apart is to consider how closed lines, also called loops, intersect on the respective surfaces. As is pirctured below in Figure 0.1, the two lines on the torus intersect only once, and no matter how you slide or distort either of these lines, they will always intersect at least once. This does not hold true for the sphere, and so we can be certain that these two objects are indeed different. Note that it is possible to choose two closed lines on the torus that do not intersect each other, but only if they go around the same way. On the sphere it is impossible to find two lines that always intersect, even after continuous sliding.

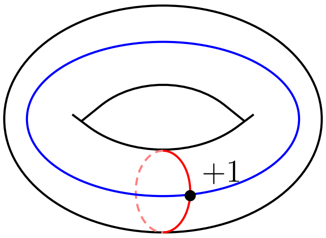

Figure 0.1: Lines intersecting on a torus

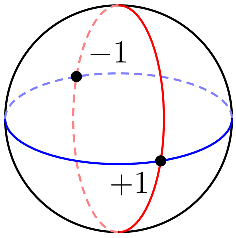

Figure 0.2: Lines intersecting on a sphere

The torus and the sphere are examples of closed, oriented 2-dimensional manifolds, and the lines that intersect on these manifolds are examples of 1-dimensional submanifolds. We can make counting intersections more useful by assigning an integer to each intersection based on the **orientation** of the lines, and of the underlying surface, assuming that we can orient the surface. We call the sum of these intersections the **intersection number**. Then, by Figure 0.1 and 0.2, we can see that the intersection number of the given lines on the torus is one, and on the sphere it is zero.

If we review how we moved the lines on the sphere before, we notice that after sliding the lines continuously, i.e. without breaking the line, the intersection number does not change. Similarly, we can see from Figure 0.3 that deforming the lines on the torus has no effect on the intersection number. This property is called **homotopy invariance**, and essential for our study of manifolds.

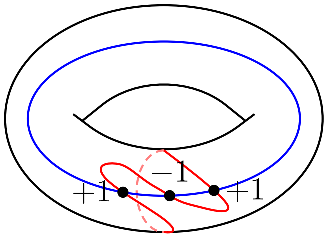

Figure 0.3: Deforming the red Line

The goal of this paper is to categorize 4-dimensional manifolds by examining how 2-dimensional submanifolds intersect on them. This is, in many ways, analogous to the above 2-dimensional case. However, one must be careful not to rely too much on one's intuition of 2 dimensions when examining 4 dimensions, since it can lead to false results. The tool that will emerge from this study, the **intersection form**, is of algebraic nature.

The reason to consider 4-dimensional manifolds is that they are special compared to manifolds in other dimensions. In dimensions 1, 2 and 3, the "low" dimensions, any two smooth manifolds that are homeomorphic are also diffeomorphic. For dimensions 5 and higher there are at most finitely many smooth structures on a given manifold. In dimension 4 there are manifolds that admit infinitely many smooth structures, for example, the so called **exotic** $\mathbb{R}^4$**'s** which are spaces that are homeomorphic, but not diffeomorphic, to $\mathbb{R}^4$. Even worse, there are actually uncountably many such spaces (up to diffeomorphism). Such exotic spaces can be constructed via **Handle Decompositions** or **h-cobordisms**, which we will not consider in this thesis. A gentle introduction to these concepts can be found in Alexandru Scorpan's **The Wild World of 4-Manifolds** [7].

Our goal will be to prove **Whitehead's Theorem on Intersection Forms**, i.e. that simply-connected, smooth, oriented 4-manifolds are determined uniquely up to homotopy equivalence by their intersection form. Indeed, a stronger result has been proven by Freedman: such manifolds are also uniquely determined up to homeomorphism by their intersection form. However, a proof of that theorem is beyond the scope of this thesis. Along the way we will stumble upon other interesting results.

To prove these wonderful statments, we will have to first review some Differential Topology. Specifically, we will prove that every element of the second homology of any 4-manifold can be represented by an embedded surface. After, we will look at some general properties of symmetric bilinear forms.

I would like to thank Will Merry for his excellent lectures and helping me understand this extraordinary topic. I also want to thank Bernd Ladwig and Dominique Heyn for proofreading my thesis.

1 Surface Representation

1.1 Poincaré Duality

To start we will review a central result in both Differential Geometry and Algebraic Topology, which we will frequently need in its different forms, is Poincaré Duality. I will state this result using the language of Differential Geometry and of Algebraic Topology separately.

Theorem 1.1.1. [6, Thm 6.4.1] *Let M be an oriented smooth m-dimensional closed manifold. Then the* ***Poincaré pairing***

$$\begin{array}{ccc} H^k_{\mathrm{dR}}(M) \times H^{m-k}_{\mathrm{dRc}}(M) & \to & \mathbb{R} \\ ([\omega],[\tau]) & \longmapsto & \displaystyle\int_M \omega \wedge \tau \end{array}$$

is non degenerate.

Remark 1.1.2. This is equivalent to the statement that the homomorphism

$$\begin{array}{cccc} \mathrm{PD}: & H^k_{dR}(M) & \longrightarrow & H^{m-k}_{dRc}(M)^* \\ & [\omega] & \mapsto & ([\tau] \longmapsto \displaystyle\int_M \omega \wedge \tau) \end{array}$$

is an isomorphism. In general, i.e. when M does not satify all the above conditions, we say that a manifold M satisfies Poincaré Duality if PD is an isomorphism. From this it follows that for any $Q \subset M$ k-dimensional closed submanifold we can define the linear form $\Lambda_Q([\omega]) := \int_Q \omega$, which, since PD is an isomorphism, gives us a unique de Rham cohomology class τ_Q, which satisfies

$$\int_M \omega \wedge \tau_Q = \int_Q \omega.$$

We say that the class τ_Q represents our surface Q. This hints at two things. First, we can consider when the opposite of the above is true: when can we represent a cohomology class by some submanifold? Secondly, the Poincaré pairing hints at the definition of the intersection form.

The above formulations were taken from Salamon's Lecture Notes: An Introduction to Differential Topology [6], see Appendix Subsection 6.2. Next we will look at the same statement in singular (co)homology, which is isomorphic to de Rham cohomology, with real coefficients.

Theorem 1.1.3. [4, Cor 39.9] *Let M be an oriented m-dimensional closed topological manifold with fundamental class $[M] \in H_m(M;\mathbb{Z})$. Then*

$$\begin{array}{ccc} H^k(M) & \longrightarrow & H_{m-k}(M) \\ \alpha & \longmapsto & \alpha \cap [M] \end{array}$$

is an isomorphism.

1.2 Smoothing Continuous Functions

As discussed in Remark 1.1.2, we can represent surfaces by cohomology classes. Now we will consider the opposite, representing homology classes by preferably smooth surfaces. To ensure that these surfaces are indeed smooth, we will need the following standard results from Differential Topology, which allow us to smooth continuous functions without changing their homotopy class.

Theorem 1.2.1 (Whitney Approximation Theorem)**.** [3, p 138] *Let N and M be smooth manifolds and $F : N \to M$ a continuous function. Then F is homotopic to a smooth map $\tilde{F} : N \to M$.*

Further, if F is already smooth on some closed subset $A \subset N$, then we can ensure that $\tilde{F}\big|_A = F\big|_A$

First, we need to prove the following Lemma:

Lemma 1.2.2. *For any continuous function $F : N \to \mathbb{R}^m$, given any continuous function $\delta : N \to \mathbb{R}_{>0}$, there exists a smooth map $\tilde{F} : N \to \mathbb{R}^m$ such that:*

$$|F(x) - \tilde{F}(x)| < \delta(x), \qquad \forall\ x \in N.$$

Moreover, if $A \subset N$ is closed and F is already smooth on A, then $\tilde{F}$ can be chosen in such a way that $\tilde{F}\big|_A = F\big|_A$.

Proof. If F is smooth on A, we can find some neighborhood U of A on which $F\big|_A$ has a smooth extension, we will call this extension F_0. If we do not have such a set A, then we set $U = A = \emptyset$. Let

$$U_0 := \{y \in U : |F_0(y) - F(y)| < \delta(y)\}$$

It is readily seen that $A \subset U_0$, since by definition $F_0\big|_A = F\big|_A$. Next we will show that we can find a countable open cover $\{U_i\}_i$ of $N\backslash A$, and points $v_i \in \mathbb{R}^m$ such that $|F(y) - v_i| < \delta(y)$ for all $y \in U_i$.

To do this, let, for all $x \in N\backslash A$, U_x be a neighborhood of x contained in $N\backslash A$, such that for all $y \in U_x$:

$$\delta(y) > \frac{1}{2}\delta(x) \qquad \text{and} \qquad |F(y) - F(x)| < \frac{1}{2}\delta(x).$$

Then, if $y \in U_x$, we have $|F(y) - F(x)| < \delta(y)$. The collection of all these U_x is a open cover of $N\backslash A$. Choose a countable subcover $\{U_{x_i}\}$, and write $U_i = U_{x_i}$ and $v_i = F(x_i)$, and so we get $|F(y) - v_i| < \delta(y)$ for all $y \in U_i$, as desired.

Now let $\{\rho_0\} \cup \{\rho_i\}_i$ be a partition of unity subordinate to the cover $\{U_0\} \cup \{U_i\}_i$ of N, and define:

$$\begin{array}{rccc} \tilde{F}: & N & \longrightarrow & \mathbb{R}^m \\ & y & \longmapsto & \rho_0(y)F_0(y) + \sum \rho_i(y)v_i. \end{array}$$

Then $\tilde{F}$ is smooth, equal to F on A, and for any $y \in N$:

$$\begin{aligned}
|\tilde{F}(y) - F(y)| &= \left|\rho_0(y)F_0(y) + \sum \rho_i(y)v_i - \overbrace{\Big(\sum_{i \geqslant 0} \rho_i\Big)}^{=1} F(y)\right| \\
&\leqslant \rho_0(y)|F_0(y) - F(y)| + \sum_{i \geqslant 1} \rho_i |v_i - F(y)| \\
&< \rho_0(y)\delta(y) + \sum_{i \geqslant 1} \rho_i(y)\delta(y) = \delta(y).
\end{aligned}$$

■

Proof of the Whitney Approximation Theorem. By the Whitney Embedding Theorem [3, p 133], we can assume that M is a submanifold of some $\mathbb{R}^m$. Let U be a tubular neighborhood of $M \subset \mathbb{R}^m$, and let $r : U \to M$ be the smooth retraction defined by $r := \pi \circ \exp^{-1}$ (See Theorem 6.3.1). For all $x \in M$ define:

$$\delta(x) = \sup\{\varepsilon \leqslant 1 \big| B_\varepsilon(x) \subset U\},$$

which is a continuous function. Let $\tilde{\delta} := \delta \circ F : N \to \mathbb{R}$. By Lemma 1.2.2, there exists a smooth map $\tilde{F} : N \to \mathbb{R}^m$ that is $\tilde{\delta}$-close to F. Define:

$$\begin{array}{rccc}
H: & N \times I & \longrightarrow & M \\
 & (p, t) & \longmapsto & r((1-t)F(p) + t\tilde{F}(p)).
\end{array}$$

This is well defined since, by definition $|\tilde{F}(p) - F(p)| < \tilde{\delta}(p) = \delta(F(p))$ for all $p \in M$, which means that $\tilde{F}(p)$ is contained in a ball of radius $\delta(F(p))$ around $F(p)$. Since this ball is contained in U, so is the line segment from $F(p)$ to $\tilde{F}(p)$.

This proves that H is a homotopy with $H_0 = F$ and $H_1 = r \circ \tilde{F}$. ■

2 Connection to the Intersection Number

2.1 Representation

Having proven that we can smooth submanifolds, let us prove the first statement hinted at by Remark 1.1.2, namely that every (co)homology class can be represented by a smooth submanifold. This statement is not true in general, however for submanifolds of 4-manifolds, in particular for surfaces, this is always possible.

The ability to go back and forth between surface representative and homology classes so easily is very powerful since it allows us to use both the geometric and algebraic tools developed in Differential Geometry and Algebraic Topology, respectively. This will be used extensively in the proof of Theorem 4.2.1, the central result of this thesis.

We assume our manifold to be simply connected, thus we get a representative immersed surface almost for free by the Hurewicz theorem. The main focus of the following proof will be to modify our immersed surface, such that it becomes an embedded one, while preserving its homology.

Lemma 2.1.1. [7, p 112] *Let M be closed, oriented and simply-connected smooth 4-manifold. Then every elemnt of $\langle a\rangle \in H_2(M;\mathbb{Z})$ can be represented by a smooth embedded surface S, i.e. if we move to de Rham cohomology we want that for all $[\omega] \in H^2_{\mathrm{dR}}(M)$: $\int_S \omega = \int_M \omega \wedge \alpha$, where α is exactly $\langle a\rangle$ modulo our choice of (co)homology theory.*

Proof. Since M is simply connected, by the Hurewicz Theorem, we can find some map $f : S^2 \to M$ which maps to $\langle a\rangle$ under the Hurewicz map. We know that f can be choosen to be smooth by the Whitney Approximation Theorem.

Further, f can be chosen as an immersion by modifying it in local coordinates. Let $\varphi : U \to B^2$ and $\psi : V \to B^4$ be coordinate charts on S^2 and M. Write $\tilde{f} : B^2 \to B^4$; $\tilde{f}(x) := \psi \circ f \circ \varphi^{-1}(x)$. We can then modify this locally by choosing some bump function

$$\rho : B^2 \to [0,1] : \quad \rho(x) = \begin{cases} 1 & x = 0 \\ 0 & |x| > \frac{2}{3}, \end{cases}$$

and a matrix $A \in \mathbb{R}^{4\times 2}$. Now define $\tilde{g} : B^2 \to B^4$ as $\tilde{g}(x) := \tilde{f}(x) + \rho(x)Ax$. Then $\tilde{g}$ is an immersion for some suitable A. By making sure that $|A|$ is small, $\tilde{f}$ and $\tilde{g}$ will be homotopic via $\tilde{F}(x,t) = (1-t)\tilde{f}(x) + t\tilde{g}(x)$. Going back to M we still have a homotopy via $F\big|_U = \varphi^{-1} \circ \tilde{F} \circ \varphi$. By choosing an atlas with a partition of unity subordinate to the open cover we can make sure that $df(x)$ is injective for all $x \in M$.

By the same method we can ensure that all self intersections of f are transverse double points, since we will be able to modify f locally if more than two points intersect. And so, since the points of the self intersection are isolated by transversality, we have at most finitely many such points.

Let $p, q \in S^2$ be such a self-intersection. That means $p \neq q$ and $f(p) = f(q)$ and there exist open neighborhoods A and B of p and q such that $f(A) \pitchfork f(B)$ at $f(p)$. Now let U be a connected, open neighborhood of M around $f(p)$, such that $f^{-1}(U) = A \sqcup B$. Choose a coordinate chart $\varphi : U \to \mathbb{R}^4 \cong \mathbb{C}^2$, such that

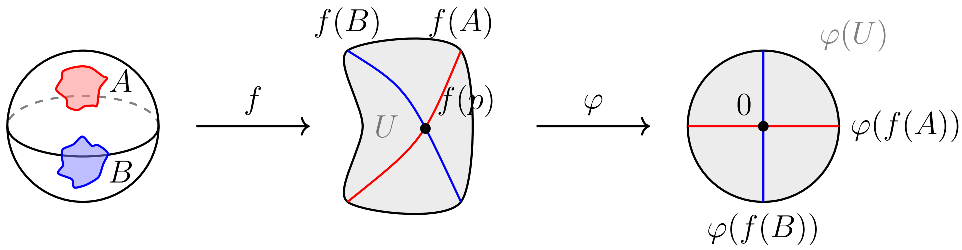

Figure 2.1: $\varphi(f(A))$ and $\varphi(f(B))$ in $\mathbb{C}^2$

$$
\begin{gathered}
\varphi(f(p)) = \varphi(f(q)) = 0 \\
\varphi(U) = B_{\mathbb{C}^2}(0,1) \\
\varphi(f(A)) = \{z \in \mathbb{C}^2 \mid z_2 = 0, |z| < 1\} \\
\varphi(f(B)) = \{z \in \mathbb{C}^2 \mid z_1 = 0, |z| < 1\}.
\end{gathered}
$$

Notice that $D_A := \varphi(f(A))$ and $D_B := \varphi(f(B))$ are two unit 2-disks. Next we will show that $\partial D_A \cup \partial D_B$ is a Hopf link [5].

First notice that ∂D_A and ∂D_B are both contained in $S^3 \subset \mathbb{C}^2$. For simplicity we will first consider "slices" of S^3. View $\mathbb{C}^2$ as $\mathbb{R}^4$ and consider:

$$S_t := S^3 \cap \{(x, y, z, t_0) \in \mathbb{R}^4 \mid t_0 = t\}.$$

Heuristically, it can be helpful to view t as a variable representing time. With little consideration it becomes clear that S_t is the 2-sphere in the first three coordinates of radius $\sqrt{1-t^2}$, and so for $|t| > 1$ the set is empty. For $t = 0$ we can see that $\partial D_A \cap S_0 = \partial D_A$, i.e. the 1-sphere in the first two coordinates. On the other hand, $\partial D_B \cap S_0 = \{(0, 0, \pm 1)\}$. For different values of $t \neq 0$ the intersection of S_t with D_A is empty and $\partial D_B \cap S_t = \{(0, 0, \pm\sqrt{1-t^2})\}$.

Next observe that:

$$\bigcup_{t\in[-1,0]}^{\cdot} S_t \cong D^3 \quad \text{and} \quad \bigcup_{t\in[0,1]}^{\cdot} S_t \cong D^3.$$

This means we can view S^3 as $D^3 \cup_{S^2} D^3$, using the identity as the attachment map. By comparing this with our slices above, we can see that ∂D_A is the equator of S^2. Our other circle ∂D_B is a line through our 3-disks from $(0, 0, -1)$ to $(0, 0, 1)$. When the 3-disks get glued together, this line forms a circle.

The linking number is exactly the intersection number $D_A \cdot \partial D_B$, which is ± 1 depending on the orientation. It follows that the loops ∂D_A and ∂D_B link exactly once making a Hopf Link. See Appendix 6.5. Knowing this, we can now resolve the self-intersection by realizing that the Hopf link is the boundary of $\varphi(f(A \sqcup B))$:

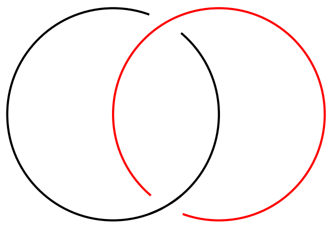

Figure 2.2: Hopf Link

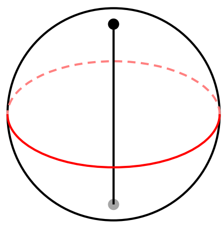

Figure 2.3: Hopf Link in S^3

$$\begin{aligned}\partial\varphi(f(A \sqcup B)) &= \partial\{z \in \mathbb{C}^2 \mid z_1 z_2 = 0, |z| \leqslant 1\} \\ &= \{z \in \mathbb{C}^2 \mid z_1 z_2 = 0, |z| = 1\} \\ &= \{z \in \mathbb{C}^2 \mid z_2 = 0, |z| = 1\} \cup \{z \in \mathbb{C}^2 \mid z_1 = 0, |z| = 1\} \\ &= \partial D_A \cup \partial D_B\end{aligned}$$

is exactly a Hopf link. We remove the disks $\varphi(f(A))$ and $\varphi(f(B))$ and replace them with

$$\Omega := \{(z_1, z_2) \in D^4 \mid z_1 z_2 = \rho(z)\},$$

where

$$\rho(z) = \begin{cases} 1 & |z| \leqslant 1/4 \\ 0 & |z| \geqslant 3/4 \end{cases}$$

is a smooth bump function. One quickly verifies that Ω is a connected, smooth and oriented 2-manifold and $\partial\Omega = \partial D_A \cup \partial D_B$. Since Ω is a manifold, it follows that the self-intersection is removed.

We can consider how the genus is affected by this operation, by first considering the Euler characteristic. From a cellular perspective, to get the new surface, we discard two 2-cells, and replace them with a new 2-cell. To do this, we also need to add a 1-cell, to attach the 2-cell along:

$$\chi_{\text{old}} - \chi_{\text{new}} = 2$$

By the formula $\chi = 2 - 2g$, it follows that the genus increased by one.

We can now repeat the whole process for all double-points, and so the resulting surface will be an embedding.

Removing the double point does not change the homology thanks to excision. More specifically, let S be the surface before removing a double point, and let S^* be the surface after. Let us also say that we only changed the surface in some ball $B \subset M$. Write $X = B \cap S$ and $X^* = B \cap S^*$. Then $\partial X = \partial X^*$, and X and X^* represent the same homology class in $H_2(B)$, since $H_2(B)$ trivial. These two facts imply that they also represent the same homology class in $H_2(B, \partial B)$.

Write $M^\circ := M \backslash B$. Then, by Mayer-Vietoris, we get:

$$0 \cong H_2(S^3) \longrightarrow H_2(M^\circ) \oplus H_2(B) \xrightarrow{\sim} H_2(M) \longrightarrow H_1(S^3) \cong 0.$$

Note that for this to work we need to expand M° and B by a bit, such that the union of their interiors gives the whole space. Since the surfaces S and S^* represent the same homology class within M°, and they represent the same class within B. By the above isomorphism it follows that, they also represent the same homology within $H_2(M)$.

■

2.2 Intersection

We will now show for general manifolds, that if surface representatives exist, their intersection number coincides exactly with the Poincaré pairing. To do this, we will first carefully construct a representative of a given submanifold, a so called Thom class.

Let $Q^l \subset M^m$ be a closed oriented submanifold of some oriented manifold without boundary. Let $TQ^\perp_\varepsilon$ be the ε-neighborhood of the zero section in the normal bundle, and let U_ε be the tubular ε-neighborhood of Q. For ε small enough we know by the Tubular Neighborhood Theorem, that the map $\exp : TQ^\perp_\varepsilon \to U_\varepsilon$ is a diffeomorphism. Now let $\tau_\varepsilon \in \Omega^{m-l}_c(TQ^\perp)$ be a Thom form such that:

$$\operatorname{supp}(\tau_\varepsilon) \subset TQ^\perp_\varepsilon, \qquad d\tau_\varepsilon = 0, \qquad \pi_*\tau_\varepsilon = 1.$$

We define $\tau_Q \in \Omega^{m-l}(M)$ as $\tau_Q := (\exp^{-1})^*\tau_\varepsilon$ on U_ε and zero everywhere else.

Proposition 2.2.1. [6, p 183] *Let $Q \subset M$ and $\tau_Q \in \Omega^{m-l}(M)$, the representative as above. Let P be a compact oriented smooth $(m-l)$-manifold without boundary and let $f : P \to M$ be a smooth map transverse to Q. Then*

$$Q \cdot f = \int_P f^*\tau_Q \overset{(*)}{=} \int_M \tau_Q \wedge \tau_P.$$

Remark 2.2.2. In the above Proposition, $(*)$ follows from Remark 1.1.2 when we assume f to be an embedding.

Proof. By assumption $f^{-1}(Q)$ is a finite set which we will denote by:

$$f^{-1}(Q) =: \{p_1, \dots, p_n\}$$

and observe that:

$$T_{f(p_i)}M = T_{f(p_i)}Q \oplus \operatorname{im} df(p_i), \qquad i = 1, \dots, n.$$

Since $\dim P + \dim Q = \dim M$, the derivative $df(p_i) : T_{p_i}P \to T_{f(p_i)}M$ is an injective linear map and hence its image inherits an orientation from $T_{p_i}P$. The intersection index $\iota(p_i; Q, f) \in \{\pm 1\}$ is obtained by comparing orientations, and, by definition, the intersection number is then the sum of these indeces:

$$Q \cdot f = \sum_{i=1}^{n} \iota(p_i; Q, f).$$

It follows from the injectivity of $df(p_i)$ that the restriction of f to a sufficiently small neighborhood $V_i \subset P$ of p_i is an embedding. Its image is transverse to Q. Choosing $\varepsilon > 0$ sufficiently small and shrinking V_i, we many assume that the V_i are pairwise disjoint and that the tubular neighborhood U_ε satisfies:

$$f^{-1}(U_\varepsilon) = \bigcup_{i \leqslant n} V_i.$$

Since $\operatorname{supp}(\tau_Q) \subset U_\varepsilon$, hence $\operatorname{supp}(f^*\tau_Q) \subset f^{-1}(U_\varepsilon) = \bigcup_{i \leqslant n} V_i$ and so:

$$\int_P f^*\tau_Q = \sum_{i=1}^{n} \int_{V_i} f^*\tau_Q = \sum_{i \leqslant n} \int_{V_i} (\exp^{-1} \circ f)^* \tau_\varepsilon \tag{1}$$

Here the second equation uses the exponential map and the Thom form $\tau_\varepsilon = \exp^* \tau_Q \in \Omega_c(TQ^\perp)$ with support in $TQ^\perp_\varepsilon$.

Now choose a local trivialization

$$\psi_i : TQ^\perp\big|_{W_i} \to W_i \times \mathbb{R}^{m-l}$$

of the normal bundle $TQ^\perp$ over a contractible neighborhood $W_i \subset Q$ of $f(p_i)$ such that the open set $TQ^\perp_\varepsilon\big|_{W_i}$ is mapped diffeomorphically onto $W_i \times B_\varepsilon$, where B_ε is the open ball of radius ε in $\mathbb{R}^{m-l}$. Let $\tau_i \in \Omega^{m-l}(W_i \times B_\varepsilon)$ be the Thom form given by

$$\psi_i^* \tau_i = \tau_\varepsilon.$$

Equation (1) gives us

$$\int_P f^*\tau_Q \overset{(1)}{=} \sum_{i \leqslant n} \int_{V_i} (\exp^{-1} \circ f)^* \tau_\varepsilon = \sum_{i \leqslant n} \int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^* \tau_i \tag{2}$$

Now consider the composition:

$$f_i := \pi'' \circ \psi_i \circ \exp^{-1} \circ f\big|_{V_i} : V_i \to B_\varepsilon.$$

If $\varepsilon > 0$ is chosen sufficiently small, this is a diffeomorphism; it is orientation preserving if $\iota(p_i; Q, f) = 1$, and orientation reversing otherwise. Since W_i is contractible, there is a homotopy $h_t : V_i \to W_i$ such that

$$h_0 \equiv f(p_i) \quad \text{and} \quad h_1 = \pi' \circ \psi_i \circ \exp^{-1} \circ f\big|_{V_i} : V_i \to W_i.$$

And so

$$h_1 \times f_i = \psi \circ \exp^{-1} \circ f\big|_{V_i} : V_i \to W_i \times B_\varepsilon.$$

Moreover, the pullback of the Thom form $\tau_i \in \Omega^n(W_i \times B_\varepsilon)$ under the homotopy $h_t \times f_i$ has compact support in $[0,1] \times V_i$. It now follows that

$$\begin{aligned}
\int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^* \tau_i &= \int_{V_i} (h_1 \times f_i)^* \tau_i && \text{(by definition)} \\
&= \int_{V_i} (h_0 \times f_i)^* \tau_i && \text{(by homotopy)} \\
&= \iota(p_i; Q, f) \int_{\{f(p_i) \times B_\varepsilon\}} \tau_i && \text{(orientation of } f\text{)} \\
&= \iota(p_i; Q, f) && \text{(by definition of } \tau_i\text{)}
\end{aligned}$$

By combining with (2), we find that:

$$\int_P f^* \tau_Q = \sum_{i \leqslant n} \iota(p_i; Q, f) = Q \cdot f.$$

■

3 Intersection Forms

3.1 Symmetric Bilinear Forms

Before considering intersection forms on manifolds, it will be useful to first familiarize ourselves with the properties of general symmetric bilinear forms.

Let us consider some finitely generated free abelian group A and let Q be some symmetric bilinear form over A.

Definition 3.1.1. [7, p 120] Let A and Q be as above.

1. The **rank** $\mathrm{rk}(Q)$ is the dimension of A.

2. The **signature** $\sigma(Q)$ of Q is the number of positive eigenvalues b^+ minus the number of negative eigenvalues b^-.

3. Q is even $Q(\alpha, \alpha) \equiv_2 0$ (called odd otherwise).

4. If for all non-zero $a \in A : Q(a, a) > 0$ we call Q **positive definite**.

5. If for all non-zero $a \in A : Q(a, a) < 0$ we call Q **negative definite**.

6. If neither of the above two definitions hold, we call Q **indefinite**.

7. The **direct sum** $Q = Q_1 \oplus Q_2$ of the forms Q_1 and Q_2 (defined on A_1, A_2 respectively) is defined on $A := A_1 \oplus A_2$ such that if $a, b \in A$ split into $a = a_1 + a_2$ and $b = b_1 + b_2$ with $a_i, b_i \in A_i$, then $Q(a, b) := Q_1(a_1, b_1) + Q_2(a_2, b_2)$.

8. If $k > 0$, then kQ denotes the k-fold sum $\bigoplus_k Q$. If $k < 0$, then $kQ := |k|(-Q)$. If $k = 0$, then $kQ := 0$ is the trivial group.

Remark 3.1.2. The signiture is additive, i.e.: $\mathrm{sign}(Q' \oplus Q'') = \mathrm{sign}\, Q' + \mathrm{sign}\, Q''$.

The following two small Lemmas will prepare us in proving that the intersection form is unimodular on the manifolds that we are considering.

Lemma 3.1.3. [1, p 10] *For any $x \in A$ define $L_x \in A^*$ by $L_x(y) = Q(x, y)$. We then get a homomorphism: $L : A \to A^*$. The form Q is unimodular if and only if L is an isomorphism.*

Proof. Fix a basis $(a_1, \ldots, a_n) \subset A$ and choose the dual basis $(a_i^*)_i \subset A^*$ (each a_i^* is defined by $a_i^*(a_j) = \delta_{ij}$). Since $L(a_i) = \sum_j Q(a_j, a_i)a_j^*$, the matrix representation of L in this basis is $B = \big(Q(a_i, a_j)\big)_{ij}$. This matrix is invertible (over $\mathbb{Z}$) if and only if $\det B = \pm 1$ which is true if and only if L is an isomorphism. ∎

Lemma 3.1.4. *Suppose that the restriction of the symmetric bilinear form Q to $B < A$ is unimodular. Then $Q := Q\big|_B \oplus Q\big|_{B^\perp}$, where $B^\perp := \{y \in A \mid Q(x, y) = 0, \forall x \in B\}$.*

Proof. If $b \in B \cap B^{\perp}$ and $b \neq 0$, then $Q(a, b) = 0, \forall a \in B$, contradicting the fact that $Q\big|_B$ is unimodular. Now for all $x \in A$ we can consider the function $b \to Q(x, b)$ on B. By unimodularity of $Q\big|_B$, and Lemma 3.1.3, there is a unique $y \in B$ such that $Q(x, b) = Q(y, b),\ \forall b \in B$.

Now $x - y \in B^{\perp}$, and so $x = b + (x - b) \in B + B^{\perp}$. And since $x \in A$ was arbitrary, we find that $A = B \oplus B^{\perp}$. The unimodularity of $Q\big|_{B^{\perp}}$ follows from the fact that $\det Q = \det Q\big|_B \cdot \det Q\big|_{B^{\perp}} = \pm \det Q\big|_{B^{\perp}}$. ■

3.2 Intersection Form

Now we can finally look at the definition of an intersection form on a manifold. I want to stress that this definition arises naturally when considering the questions that come up in Remark 1.1.2.

Definition 3.2.1. [1, p 7] Let M be a closed, simply connected, oriented 4-manifold. The symmetric bilinear form

$$\begin{array}{rccc} Q_M : & H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ & (\alpha, \beta) & \longmapsto & (\alpha \cup \beta) \cap [M] \end{array}$$

is called the intersection form of M.

Remark 3.2.2. By the above defintion we can immediately make some observations:

1. Due to Poincaré duality we can write:

$$\begin{array}{rccc} Q_M : & H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ & (a, b) & \longmapsto & (a^* \cup b^*) \cap [M]. \end{array}$$

2. By the equivalence of de Rham and Singular Cohomology, it follows that

$$Q_M(a, b) = \int_M \alpha \wedge \beta$$

where α and β are the representative 2-forms of a and b, respectively.

3. Since Q_M is bilinear, it vanishes on torsion elements, i.e. if a or b is a torsion element, then $Q_M(a, b) = 0$. This means that we can choose a basis for the free part of $H^2(M;\mathbb{Z})$ and represent Q_M by a matrix of determinant ± 1. [7, p 120]

4. By changing the orientation of M we change the signature:

$$\text{sign } Q_{\overline{M}} =: \text{sign } \overline{M} = -\text{sign } M.$$

5. Also, by Theorem 4.1.1 and Remark 3.1.2, we get that

$$\text{sign}(M \# N) = \text{sign } M + \text{sign } N.$$

Example 3.2.3. Consider the space $S^2 \times S^2$. To calculate the intersection form $Q_{S^2 \times S^2}$ we first have to calculate the structure of the cohomology ring.

$S^2 \times S^2$ can be constructed by a 0-cell, two 2-cells, and a 4-cell. So we get the following group structure:

$$H^0(S^2 \times S^2) = \mathbb{Z}, \qquad H^2(S^2 \times S^2) = \mathbb{Z}^2, \qquad H^4(S^2 \times S^2) = \mathbb{Z}$$

with all other homologies being zero. The ring structure of $H^\bullet(S^2 \times S^2)$ can be found by the Künneth Formula:

$$H^\bullet(S^2 \times S^2) \cong H^\bullet(S^2) \otimes H^\bullet(S^2) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2)$$

where α and β are the generators of the two $H^2(S^2)$.

From this we can see that the intersection form is given by:

$$Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From the matrix form the following properties follow from definition. First $Q_{S^2 \times S^2}$ is even, but indefinite. Also the signature $\sigma(S^2 \times S^2) = 0$.

Lemma 3.2.4. *Let M be a simply connected, closed smooth 4-manifold. Then the intersection form Q_M is unimodular, i.e.* $\det Q_M = \pm 1$. [7]

Proof. By the dual coefficients theorem and our assumption that M is simply connected, we find that $H^2(M;\mathbb{Z})$ is a free $\mathbb{Z}$-module. By Lemma 3.1.3 we know that Q_M is unimodular if and only if the map

$$\begin{array}{rccc} \hat{Q}_M: & H^2(M;\mathbb{Z}) & \longrightarrow & \operatorname{Hom}(H^2(M;\mathbb{Z}),\mathbb{Z}) \\ & \alpha & \longmapsto & x \mapsto Q_M(\alpha, x) \end{array}$$

is an isomorphism. We will argue that this map coincides with the Poincaré duality morphism, which we know to be an isomorphism, given by

$$\begin{array}{ccc} H_2(M;\mathbb{Z}) & \xrightarrow{\sim} & H^2(M;\mathbb{Z}) \\ \alpha & \longmapsto & \alpha^* \end{array}$$

where $\alpha^* \cap [M] = \alpha$. Since torsion elements vanish in Q_M we can assume $H_2(M;\mathbb{Z})$ to be free. By the dual universal coefficient theorem we have the following isomorphism

$$\begin{array}{ccc} H^2(M;\mathbb{Z}) & \xrightarrow{\sim} & \operatorname{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \\ \alpha^* & \longmapsto & x \mapsto \alpha^* \cap x. \end{array}$$

By Poincaré Duality we get the isomorphism: $H_2(M;\mathbb{Z}) \xrightarrow{\sim} \operatorname{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z})$. By general properties of the cap product we know

$$Q_M(\alpha^*, \beta^*) = (\alpha^* \cup \beta^*) \cap [M] = \beta^* \cap (\alpha^* \cap [M]) = \beta^* \cap \alpha =: Q_M(\alpha, \beta^*)$$

and so the above isomorphism coincides with $\hat{Q}_M$. ∎

4 Main Results

4.1 Intersection Forms and Connected Sums

Before proving Whitehead's theorem, we will first present a small result which can be useful when calculating intersection forms or constructing manifolds with certain intersection forms.

We will need the **connected sum**, which, intuitively speaking, is the "simplest way" of combining two manifolds of the same dimension, M and N, into one which we write as $M\#N$.

More rigorously, by [7, p 117], we choose a small open $m-$ball in each manifold and remove it. We call the remaining two manifolds M° and N°. Next we embed $S^{n-1} \times [0,1]$ as "collars" to the boundary, i.e. the $(m-1)$-sphere. $S^{n-1} \times 1$ gets glued onto ∂M°, and $S^{n-1} \times [0,1)$ is sent to the interior of M°. Similarly for N, $S^{n-1} \times 0$ gets identified with ∂N° and $S^{n-1} \times (0,1]$ is sent to the interior. We then identify the collars $S^{n-1} \times [0,1]$ in the obvious way to obtain $M\#N$, this also forces the identification map to be orientation reversing.

Theorem 4.1.1. *If M and N are closed, simply connected 4-manifolds, then:*

$$Q_{M\#N} = Q_M \oplus Q_N$$

Proof. This follows from the Mayer-Vietoris Sequence. By slight abuse of notation we write $N, M \subset M\#N$. We then have that $M \cap N = S^3$. Hence:

$$0 \cong H_2(S^3) \longrightarrow H_2(M) \oplus H_2(N) \longrightarrow H_2(M\#N) \longrightarrow H_1(S^3) \cong 0.$$

■

4.2 Whitehead's Theorem on Intersection Forms

It is obvious that if two 4-manifolds are homotopy-equivalent, they have the same intersection form, but it is not clear if the opposite assertion holds. Because we assumed our manifolds to be simply connected, the first and the third cohomology groups vanish (thanks to Poincaré Duality), and so $H_2(M) \cong H^2(M) \cong \mathrm{Hom}(H_2(M), \mathbb{Z})$ has no torsion, due to the Dual Universal Coefficients Theorem [4, Cor 29.9]. Thus it reasonable to conjecture that Q_M could contain all the information about M we need to identify it up to homotopy-equivalence. This is exactly what Whitehead's Theorem on Intersection Forms tells us, which classifies all simply connected topological 4-manifolds up to homotopy equivalence by their intersection form.

Theorem 4.2.1 (Whitehead's Theorem on Intersection Forms)**.** [7, p 140] *The simply connected, closed, topological 4-manifolds M and N are homotpoy equivalent if and only if $Q_M \cong Q_N$.*

Proof. This proof has two broad steps, first showing that M is homotopy equivalent to a simpler space that we construct. Second, showing that this space is only determined by the intersection form of M.

By simple-connectedness and Poincaré Duality, we know that $H^1(M) = H^3(M) = 0$. Further, by Hurewicz's Theorem, we find that $H_2(M) \cong \pi_2(M)$. Since M is simply connected and has no torsion, it is isomorphic to some $\bigoplus_m \mathbb{Z}$. Thus we can find a map:

$$f : S^2 \vee \cdots \vee S^2 \longrightarrow M$$

which induces the isomorphism $\pi_2(f) : \pi_2(S^2 \vee \cdots \vee S^2) \xrightarrow{\sim} \pi_2(M) \cong H_2(M)$. This map induces an isomorphism on all homologies but the fourth. This can be solved by considering $M^\circ = M \backslash 4\text{-ball}$, which has no H_4 by:

$$0 \longrightarrow \overbrace{H_4(\partial X)}^{\cong 0} \longrightarrow H_4(X) \longrightarrow \overbrace{H_4(X, \partial X)}^{\cong \mathbb{Z}} \longrightarrow \overbrace{H_3(\partial X)}^{\cong \mathbb{Z}} \longrightarrow 0$$

where $X := M \backslash 4\text{-ball}$. By Theorem 6.4.1, we can see that M° is homotopy equivalent to $S^2 \vee \cdots \vee S^2$.

Since M can be constructed from M° by gluing a 4-ball to M°, we deduce, by Theorem 6.4, that a space with the same homotopy type can be constructed by gluing a 4-ball B^4 to $\bigvee_m S^2$. It follows that

$$M \approx \bigvee_m S^2 \cup_\varphi D^4$$

for some suitable attaching map $\varphi : \partial D^4 \to \bigvee_m S^2$. The homotopy type of M is completely determined by the homotopy class of φ, which we will view as an element of $\pi_3(\bigvee_m S^2)$. Moreover, the fundamental class $[M] \in H_4(M)$ corresponds to the class of the attached 4-ball $[D^4]$, since by definition $[M]$ is a generator of $H_4(M)$ and by [4, Cor 20.9], not having cells of dimension 3 or 5 implies that $[D^4]$ also generates $H_4(M)$. Now we start step 2, showing that the homotopy class of φ is completely determined by the intersection form of M.

Now think of each S^2 as a copy of $\mathbb{C}P^1 \subset \mathbb{C}P^\infty$. For a quick reminder of the properties of $\mathbb{C}P^\infty$, see Appendix 6.6. Now embed

$$\bigvee_m S^2 \subset \bigtimes_m \mathbb{C}P^\infty$$

and consider the sequence

$$\pi_4\big(\bigtimes_m \mathbb{C}P^\infty\big) \to \pi_4\big(\bigtimes_m \mathbb{C}P^\infty, \bigvee_m S^2\big) \to \pi_3\big(\bigvee_m S^2\big) \to \pi_3\big(\bigtimes_m \mathbb{C}P^\infty\big). \tag{1}$$

Since $\mathbb{C}P^\infty$ is an Eilenberg-MacLane $K(\mathbb{Z}, 2)$-space, the only non-trivial homotopy group of $\times_m \mathbb{C}P^\infty$ is π_2, and so by (1) we get an isomorphism:

$$\pi_4\big(\bigtimes_m \mathbb{C}P^\infty, \bigvee_m S^2\big) \cong \pi_3\big(\bigvee_m S^2\big). \tag{2}$$

The above π_4 consists of maps $D^4 \to \times_m \mathbb{C}P^\infty$ that take the boundary $\partial D^4 \cong S^3$ to $\bigvee_m S^2$. Consequently, the isomorphism in equation (2) associates to each $\varphi : S^3 \to \bigvee_m S^2$ the class of an extended map:

$$\tilde{\varphi} : D^4 \longrightarrow \bigtimes_m \mathbb{C}P^\infty.$$

Further, since the inclusion $\bigvee_m S^2 \subset \bigtimes_m \mathbb{C}P^\infty$ induces an isomorphism on π_2, we can see by a different part of the exact sequence seen in equation (1), that for both π_2 and π_3 the pair $(\bigtimes_m \mathbb{C}P^\infty, \bigvee_m S^2)$ must vanish. By Hurewicz's theorem we can see that:

$$\pi_4\big(\bigtimes_m \mathbb{C}P^\infty, \bigvee_m S^2\big) \cong H_4\big(\bigtimes_m \mathbb{C}P^\infty, \bigvee_m S^2\big)$$

Through this identification, the class of $\tilde{\varphi}$ in π_4 is sent to the class

$$\tilde{\varphi}_*[D^4] \in H_4\big(\bigtimes_m \mathbb{C}P^\infty, \bigvee_m S^2\big) \cong H_4\big(\bigtimes_m \mathbb{C}P^\infty\big).$$

where $\tilde{\varphi}_*$ is the map induced by the map $\tilde{\varphi}$ by homology. Here the second equivalence holds because the second and third homology of $\bigvee_m S^2$ vanishes. Additionally, due to the lack of torsion we have a natural duality:

$$H^4\big(\bigtimes_m \mathbb{C}P^\infty\big) = \mathrm{Hom}\big(H_4(\bigtimes_m \mathbb{C}P^\infty),\ \mathbb{Z}\big).$$

Hence, in order to determine $\tilde{\varphi}_*[D^4]$, it is sufficient to evaluate all classes $\alpha \in H^4$ on it. Put differently, our class $\varphi \in \pi_3(\bigvee_m S^2)$ (and so the homotopy type of M) is completely determined by the values $\alpha_k(\tilde{\varphi}_*(D^4))$ for some basis $\{\alpha_k\}_k$ of $H^4(\times_m \mathbb{C}P^\infty)$

One way to obtain such a basis is by cupping the classes representing each S^2, thus we get

$$H^4\big(\bigtimes_m \mathbb{C}P^\infty\big) = \mathbb{Z}\{\omega_i \cup \omega_j\}_{i,j}$$

where ω_k denotes the 2-class dual to $[\mathbb{C}P^1]$ inside the k^{th} copy of $\mathbb{C}P^\infty$. Since

$$H^2\big(\bigtimes_m \mathbb{C}P^\infty\big) \cong H^2\big(\bigvee_m S^2\big) \cong H^2(M^\circ) \cong H^2(M),$$

we can see that each class ω_k can be viewed as a 2-class w_k of M. More specifically, the inclusion $\iota : \bigvee_m S^2 \hookrightarrow \bigtimes_m \mathbb{C}P^\infty$ extends via $\tilde{\varphi}$ to

$$M \approx \bigvee_m S^2 \cup_\varphi D^4 \xrightarrow{\iota+\tilde{\varphi}} \bigtimes_m \mathbb{C}P^\infty \quad \text{where} \quad \iota + \tilde{\varphi} = \begin{cases} \iota & p \in \bigvee_m S^2 \\ \tilde{\varphi} & p \in D^4 \end{cases}$$

The pullbacks $w_k = (\iota+\tilde{\varphi})^*\omega_k$ make a basis of $H^2(M)$. Now evaluating $\omega_i \cup \omega_j$ on $\tilde{\varphi}_*[D^4]$ in $\bigtimes_m \mathbb{C}P^\infty$ yields the same result as pulling ω_i and ω_j back to M, cupping there, and then evaluating on $[D^4]$:

$$\begin{aligned}(\omega_i \cup \omega_j)(\tilde{\varphi}_*[D^4]) &= \big((\iota + \tilde{\varphi})^*(\omega_i \cup \omega_j)\big)[D^4] \\ &= \big((\iota + \tilde{\varphi})^*\omega_i\big) \cup \big((\iota + \tilde{\varphi})^*\omega_j\big)[D^4] \\ &= (w_i \cup w_j)[D^4].\end{aligned}$$

As we have mentioned before, the class $[D^4]$ is exactly the fundamental class $[M]$ of M, it follows that

$$(w_i \cup w_j)[D^4] = Q_M(w_i, w_j).$$

Since $\{w_1, \ldots, w_m\}$ is a basis of $H^2(M)$, we get the whole intersection form. On the other hand, as we have argued, by staying in $\bigtimes_m \mathbb{C}P^\infty$ and evaluating all the $(\omega_i \cup \omega_j)$ on $\tilde{\varphi}_*[D^4]$ we fully determine the class of φ in $\pi_3\big(\bigvee_m S^2\big)$, and thus determine the homotopy type of M.

■

Example 4.2.2. [7, p 124] The complex projective plane $\mathbb{C}P^2$ has intersection form $Q_{\mathbb{C}P^2} = (+1)$, since $H_2(\mathbb{C}P^2) = \mathbb{Z}$. The second homology is generated by $[\mathbb{C}P^1]$, which can easily be seen by cellular homology. By Remark 3.2.2 we can see that $Q_{\overline{\mathbb{C}P^2}} = (-1)$.

Let us now consider $S^2 \tilde{\times} S^2$, the unique non-trivial sphere bundle over S^2. Uniqueness follows since our S^2-bundle over $S^2 = D_1^2 \cup D_2^2$ is described by the equitorial gluing map $S^1 \to SO(3)$. This is sufficient because bundles over D^2, a contractible space, are trivial. By $\pi_1(SO(3)) = \mathbb{Z}_2$ it follows that there are only two topologically distinct sphere-bundles over S^2.

Viewing D^2 as a hemisphere, we can construct $S^2 \tilde{\times} S^2$ by gluing two copies of $D^2 \times S^2$ together along the equator of the base-sphere, using the identification of S^2-fibres that rotates them once as we travel along the equator. The intersection form is:

$$Q_{S^2 \tilde{\times} S^2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

By a simple change of basis, we can see that:

$$Q_{S^2 \tilde{\times} S^2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = [-1] \oplus [1] = Q_{\overline{\mathbb{C}P^1}} \oplus Q_{\mathbb{C}P^1}.$$

By the above theorem we see that $S^2 \tilde{\times} S^2 \cong \mathbb{C}P^1 \# \overline{\mathbb{C}P^1}$.

5 Outlook

My aim of this last section is to give an outlook on the on the world of 4-manifolds.

To start with, we will look at the reverse statement of Theorem 4.1.1, which holds for 4-manifolds.

Theorem 5.0.1. [7, p 119] *If M is simply connected and Q_M splits into a direct sum $Q_M = Q_1 \oplus Q_2$, then there exists topological 4-manifolds N_1 and N_2, with intersection forms Q_1 and Q_2, such that $M = N_1 \# N_2$*

Another small statement about intersection forms on 4-manifolds is the following.

Lemma 5.0.2. *If M^4 is the boundary of some oriented 5-manifold W^5, then*

$$\text{sign } Q_M = 0.$$

It seems in line with general facts about manifolds we know, such as the fact that the degree of a map restricted to the boundary is always zero, or that the intersection number of a map restricted to the boundary with a submanifold is always zero. Having said that, for 4-manifolds the inverse of the above statement holds as well.

Theorem 5.0.3 (V. Rokhlin)**.** *If a smooth oriented 4-manifold M has:*

$$\text{sign } Q_M = 0$$

then there exists a smooth oriented 5-manifold W such that $\partial W = M$.

To finish our small outlook on the world of 4-manifolds, we consider Freedman's Classification Theorem.

Theorem 5.0.4 (Freedman's Classification Theorem)**.** *For any integral symmetric unimodular form Q, there is a closed simply-connected topological 4-manifold that has Q as its intersection form.*

- *If Q is even, there is exactly one such manifold*
- *If Q is odd, there are exactly two such manifolds, at least one of which does* not *admit a smooth structure.*

Corollary 5.0.5. *If M and N are smooth, simply-connected, and have isomorphic intersection forms, then M and N must be h-cobordant.*

Corollary 5.0.6. *If M and N have isomorphic intersections forms, then they are homeomorphic.*

In this thesis we have only considered manifolds up to homeomorphism. One can also consider 4-manifolds up diffeomorphism, this study is very different and generally more difficult. For example it is still an open question how many different smooth structures, up to diffeomorphism, exist on S^4. As is explained by in the aforementioned **The Wild World of 4-Manifolds** [7].

6 Appendix

6.1 Fundamental Classes

The fundamental class of a manifold M^m is a generator of the homology group $H_m(M;\mathbb{Z}) \cong \mathbb{Z}$. The fundamental class can be thought of as the orientation of the manifold. If M is disconnected, a fundamental class is the direct sum of the fundamental classes for each connected component.

6.2 Poincaré Duality

The following is take from Salamon's Differential Geometry Lecture Notes [6, p 182].

Lemma 6.2.1. *Here τ_Q is defined in Subsection 2.2. Let $Q \subset M$ and $\tau_Q \in \Omega^l(M)$, then*

$$\int_M \omega \wedge \tau_Q = \int_Q \omega$$

for every $(m-l)$ form $\omega \in \Omega^{m-l}(M)$

Proof. Denote the inclustion of the zero section in $TQ^\perp$ by

$$\iota_Q : Q \longrightarrow TQ^\perp.$$

For every closed form $\omega \in \Omega^{m-l}(M)$ we compute:

$$\begin{aligned} \int_M \omega \wedge \tau_Q &= \int_{U_\varepsilon} \omega \wedge \tau_Q && (1) \\ &= \int_{TQ_\varepsilon^\perp} \exp^* \omega \wedge \tau_\varepsilon && (2) \\ &= \int_Q \iota_Q^* \exp^* \omega && (3) \\ &= \int_Q (\exp \circ \iota_Q)^* \omega && (4) \\ &= \int_Q \omega. && (5) \end{aligned}$$

Here, (3) follows from τ_ε being a Thom class and $TQ_\varepsilon^\perp \subset TQ^\perp$ being a star shaped open neighborhood of the zero section. Step (5) follows from the fact that:

$$\exp \circ \iota_Q : Q \longrightarrow M$$

is just the inclusion. ■

6.3 Tubular Neighborhood Theorem

Theorem 6.3.1. *Let N be a Riemannian n-manifold without boundary, let $Q \subset N$ be a compact m-dimensional submanifold without boundary, and let $\varepsilon_Q := \inf_{q \in Q} \operatorname{inj}(q, N) > 0$. For $\varepsilon < \varepsilon_Q$ let:*

$$TQ_\varepsilon^\perp := \big\{(q,w) \in TQ^\perp \;\big|\; |w| < \varepsilon\big\} \quad \textit{and} \quad U_\varepsilon\big\{p \in N \;\big|\; \inf_{q \in Q} d(p,q) < \varepsilon\big\}.$$

Then

$$\begin{array}{ccc} TQ_\varepsilon^\perp & \longrightarrow & U_\varepsilon \\ (q,w) & \longmapsto & \exp_q(w) \end{array}$$

is a diffeomorphism.

6.4 Whitehead's Theorem

The following formulation of Whitehead's Theorem is taken from Hatchers excelent book Algebraic Topology [2, p 346].

Theorem 6.4.1 (Whitehead's Theorem)**.** *If a map $f : X \to Y$ between two cell complexes induces an isomorphism*

$$f_* : \pi_n(X) \to \pi_n(Y)$$

for all n, then f is a homotopy equivalence. If $f : X \hookrightarrow Y$ is the inclusion, then X is a deformation retract of Y.

The proof of will follow easily from the following technical lemma.

Lemma 6.4.2 (Compression Lemma)**.** *Let (X, A), (Y, B) be pairs of cellular complexes such that $B \neq \emptyset$. For each n that $X \backslash A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic* $\operatorname{rel} A$ *to a map $X \to B$.*

Proof. For $n = 0$ the condition $\pi_n(Y, B, y_0) = 0$ is equivalent to saying that (Y, B) is 0-connected. Now assume inductively that f has already been homotoped to take the skeleton X^{k-1} to B. If Φ is the characteristic map of a cell e^k of $X \backslash A$, the composition $f \circ \Phi : (D^k, S^{k-1}) \to (Y, B)$ can be homotoped into $B \operatorname{rel} S^{k-1}$ since $\pi_k(Y, B, y_0) = 0$ if $k > 0$ or (Y, B) 0-connected for $k = 0$. This homotopy of $f \circ \varphi$ induces a homotopy rel X^{k-1} of f on the quotient space $X^{k-1} \cup e^k$ of $X^{k-1} \,\dot\cup\, D^k$.

Doing this for all k-cells of $X \backslash A$ simultaneously, and taking the constant homotopy on A, we obtain a homotopy $f\big|_{X^k \cup A}$ to a map which maps into B. This extends to a homotopy on all of X.

Finitely many applications of the induction step finish the proof if the cells of $X \backslash A$ are of bounded dimension. In general we perform the homotopy of the induction step in the interval $[1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}]$. Any finite skeleton X^k is eventually stationary under these homtpies, thus we get a well defined map f_t such that $f_1(X) \subset B$. ∎

Proof of Whitehead's Theorem. Let us start with the case that f is the inclusion of a subcomplex. Consider the long exact sequence of homotopy groups for the pair (Y, X):

$$\cdots \to \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Y, X) \longrightarrow \pi_{n-1}(X) \to \dots$$

Since $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n, the relative groups $\pi_n(Y, X)$ are all zero. By applying the Compression Lemma on the identity $(Y, X) \to (Y, X)$ we get a deformation retract of Y on X.

Now let us consider the general case. We first define the mapping cylinder C_f as $(X \times I \cup Y)/\sim$, where $\sim$ is given by $(x, 1) \sim f(x)$. We can see that C_f contains both Y and $X = X \times \{0\}$ as subspaces. Moreover, we can see that Y is a deformation retract of C_f, which makes these two spaces homotopy equivalent. So all that is needed to complete this proof is to show that X and C_f are homotopy equivalent, which will be done by showing that X is a deformation retract of C_f.

By the Cellular Approximation Theorem, we can assume f to be cellular, taking the n-sekelton of X to the n-skeleton of Y, for all n. Then (C_f, X) is a pair of a cellular complexes. Then we are done by the first paragraph (since $X \hookrightarrow C_f$ is the inclusion). ■

6.5 Linking Numbers and Homotopy Theory

Definition 6.5.1. A **knot** is a (topological) embedding of S^1 into $\mathbb{R}^3$ or S^3. The reason one might consider $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ is that it is compact. A **link** is a collection of knots which do not intersect.

Definition 6.5.2. An **ambient isotopy** of M^m is a smooth map:

$$F : [0, 1] \times M \longrightarrow M$$

such that $F_0 = \mathrm{id}$.

Definition 6.5.3. Two knots are called **equivalent** if they are invariant under ambient isotopy. If $k, l : S^1 \to S^3$ are knots, and F an ambient isotopy, then k and l are equivalent if $k \circ F_1 = l$.

Any embedding k that can be ambient isotoped to the **standard unknot** $S^1 \subset \mathbb{R}^3$ is called an **unknot**.

Definition 6.5.4. The **linking number** of two knots: $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ is one of the following equivalent definitions:

1. The first definition is the one Gauss discovered:
$$\mathrm{link}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\dot\gamma_1(s), \dot\gamma_2(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|^3} ds\ dt$$
2. A definition using intersection theory. Let $\Sigma : D^2 \to \mathbb{R}^3$ such that $\partial\ \mathrm{im}\Sigma = \mathrm{im}\gamma_1$, then:
$$\mathrm{link}(\gamma_1, \gamma_2) = \Sigma \cdot \gamma_2$$

3. The linking number can also be defined as the number of crossings in the representative diagram counted with sign divided by two.
4. More generally, for any two disjoint compact, oriented manifolds $M^m, N^n \subset \mathbb{R}^{k+1}$ without boundary and total dimension $n+m=k$, the linking number is the degree of the map
$$\lambda : M \times N \to S^k, \qquad \lambda(x,y) := \frac{x-y}{||x-y||}.$$

The Hopf-Link is the link that consists of two unknots with linking number ± 1.

6.6 The Complex Projective Space $\mathbb{C}P^\infty$

To construct $\mathbb{C}P^\infty$ we take the direct limit of the spaces $\mathbb{C}P^n$ as defined by cellular homology. We know that $\pi_k(S^n) = 0$ for $k < n$. Hence for $k \to \infty$ we can see that all homotopy groups for S^∞ are zero, and it also has a cell structure. Thus, by Whitehead's Theorem 6.4, S^∞ is contractable.

We can write S^∞ as a fibre bundle of $\mathbb{C}P^\infty$:
$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty.$$

By the homotopy sequence for Fibrations we can see that $\mathbb{C}P^\infty$ is indeed the Eilenberg-MacLane $K(\mathbb{Z}, 2)$.

6.7 Assumptions, Prerequisites and Notation

This thesis requires the reader to be familiar with some of the fundamental results of Differential Geometry and Algebraic Topology. This includes the equivalence of simplicial and de Rham cohomology, cellular decompositions and Poincaré Duality. Some of the notation used can be found in the list below:

- B^n is the n-dimensional open unit ball.
- D^n is the n-dimensional unit disk (i.e. $(D^n)^o = B^n$ and $\overline{B^n} = D^n$).
- S^n is the n-dimensional sphere.
- H^k_{dR} is the k^{th} de Rham cohomology.
- H^k_{dRc} is the k^{th} compactly supported de Rham cohomology.
- For any manifold M I will denote its fundamental class by $[M]$.

References

[1] R.E. Gompf and A. Stipsicz. *4-manifolds and Kirby Calculus*. Graduate studies in mathematics. American Mathematical Society, 1999.

[2] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[3] John M. Lee. *Introduction to Smooth Manifolds*. 2000.

[4] Will J. Merry. *Algebraic Topology Lecture Notes*. Faculty from ETHZ, 2018.

[5] Mark Powell. Removing intersections of immersed surfaces in a 4-manifold, 2009.

[6] Joel W. Robbin and Dietmar A. Salamon. *Introduction to Differential Topology*. June 2018.

[7] A. Scorpan. *The Wild World of 4-manifolds*. American Mathematical Society, 2005.