Large Components in the Random Connection Model

submitted by

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I dedicate this thesis to my wonderful partner-in-everything Jolie.

You dream for both of us.

Summary

We explore the Marked Random Connection Model (MRCM) in the subcritical and supercritical regimes. The behavior of Large Components will be used to guide our exploration.

In the subcritical regime we show that large components occupy a vanishing fraction of the observation window. We do this by studying the correlation length, which describes the tail behavior of components, and is an object of interest in its own right.

In the supercritical regime we show that the largest component occupies a strictly positive fraction of the observation window. Our analysis will require various 'uniqueness' statements, which ensure that distant clusters are indeed connected. We prove the Grimmett-Marstrand theorem for the MRCM to help us sharpen our uniqueness statements, which we can then employ to prove the desired result about large components.

Contents

Notation and Terminology

Symbol	Description
$\mathbb{R}$	The set of real numbers.
$\mathbb{N}$	The set of natural numbers (including 0).
$\mathbb{Z}$	The set of integers.
$\mathbb{R}_{\geq 0}$	The set of positive elements of $\mathbb{R}$.
$[\![a, b]\!]$	The collection of integers $\{a, a+1, \dots, b-1, b\}$.
$\mathbb{P}$	The probability measure.
$\mathbb{E}$	The expected value.
$ppp(X, \mu)$	The law of the Poisson point process on a space X with intensity measure μ.
η	Standard symbol for a Poisson point process.
ψ	Symbol used for the connection function.
$\mathbf{1}\{\cdot\}$	Indicator function.
$(\mathbb{M}, \rho)$	The mark space with the associated probability measure.
$\mathbb{X}$	The space $\mathbb{R}^d \times \mathbb{M}$.
$\mathbb{X}_M$	The space $\mathbb{R}^d \times M$, for some subset $M \subseteq \mathbb{M}$.
Λ_t	The box $[-t, t]^d \times \mathbb{M}$ with side-length $2t$.
$\overline{\Lambda}_t$	The box $[-t, t]^d$ with side-length $2t$.
B_r	The ball of radius $r \in \mathbb{R}_{\geq 0}$.
Z_ψ	$\iiint_{\mathbb{R}^d \times \mathbb{M}^2} \psi(y; a, b) \mathrm{d}y \rho^{\otimes 2}(\mathrm{d}(a, b))$. λZ_ψ is the expected number of neighbors for a typical vertex.
Z_ψ^∞	$\operatorname{ess\,sup}_{a \in \mathbb{M}} \iint_{\mathbb{R}^d \times \mathbb{M}} \psi(y; a, b) \mathrm{d}y \rho^{\otimes 2}(\mathrm{d}b)$.
$\xi[\eta]$	The random connection model driven by η. Written simply as ξ when η is clear from context.
$\xi[\eta, \eta']$	The random connection model with addition edges between η and η', but not η' and itself.

Symbol	Description
$\xi[\eta;\eta']$	The random connection model $\xi[\eta]$ with η' overlaid, but no additional edges.
$\mathcal{C}(x,\xi)$	The connected component of x in ξ. We also write $\mathcal{C}_x$ when ξ is clear from context.
$\theta(\lambda)$	The probability that an inserted point at the origin o is connected to the infinite cluster. We refer to it as the 'percolation probability'.
λ_c	The critical intensity. The infimum over all λ such that $\theta(\lambda) > 0$.
$\tau(x,y)$	The probability that two inserted points x and y belong to the same cluster in $\xi[\eta^{x,y}]$. Also referred to as the 'two-point function'.
$\overline{\tau}(x,y)$	The probability that the inserted points x and y connect to the same cluster in $\xi[\eta]$. Also referred to as the 'restricted two-point function'.
$\partial^{\mathrm{in}}K$	The internal boundary of the set $K \subseteq \mathbb{R}^d$ given by $\partial^{\mathrm{in}}K := \{x \in K \mid d(x,K^c) \leq 1\}$.
$\partial^{\mathrm{ext}}K$	The external boundary of the set $K \subseteq \mathbb{R}^d$ given by $\partial^{\mathrm{ext}}K := \{x \in K^c \mid d(x,K) \leq 1\}$.

Chapter 1

Introduction

1.1 History & Motivation

Random spatial models serve as fundamental frameworks for understanding a diverse range of phenomena, from epidemiological spread and material phase transitions to the structure of social networks and communication systems. The common thread amongst these models is the emergence of non-trivial large-scale behavior arising solely from local (random) rules of interaction. A particularly significant emergent behavior is the phase transition, where only a small change in parameters lead to dramatic shifts in global system properties.

The first rigorous study of such models dates back to Ernst Ising. In his PhD thesis in 1925, Ising considered a model of ferromagnetism, where up and down 'spins'[1] are placed on the one-dimensional lattice $\mathbb{Z}$, with a temperature parameter T controlling the interaction strength. Ising proved that in one dimension there is no phase transition. He incorrectly conjectured that his model displays no phase transition in *any* dimension, a claim later disproven by Peierls in 1936 [Pei36] for the two-dimensional case. In particular, it holds true that for all dimensions $d \geq 2$ there exists a critical temperature $T_c > 0$ which divides the parameter-space into a supercritical region where magnetization occurs and a subcritical region where magnetization does not occur[2].

The first rigorous mathematical treatment of a variety of percolation systems was done by Broadbent and Hammersley in 1957 [BH57]. This is, in particularly, the first paper

[1] Each spin was represented as a $+1$ or -1.

[2] Conventionally, statistical physicists work with inverse temperature $\beta = 1/T$. In addition to being mathematically convenient it helps align terminology: $\beta > \beta_c$ refers to supercritical and vice versa.

to consider what we now call Bernoulli bond percolation. Instead of temperature this model considers a graph whose edges (or bonds) are open with some probability $p \in [0, 1]$. In their ground-breaking paper they relate percolation to self-avoiding random walks and use this to show that a non-trivial critical parameter $p_c \in (0, 1)$ exists. The Bernoulli bond percolation was extremely important as a model as it is possibly the simplest geometric model exhibiting a phase transition.

Continuing this trajectory, Gilbert [Gil61] was the first to investigate continuum percolation models with mathematical rigor in 1961. Gilbert introduced what is now known as the Gilbert disk model, where points are randomly distributed in a plane according to a Poisson point process with intensity λ, and pairs of points are connected if they lie within a specified distance of each other. Gilbert proves the existence of a critical intensity on the plane, and thus a phase transition. Further, he manages to bound the critical value from below using arguments from branching process theory.

The first generalization to what we now call the Random Connection Model (RCM) was first considered by Penrose [Pen91] in 1991. Unlike the Gilbert model, where connections between points are deterministic given their relative Euclidean distances, the RCM allows for extra randomness to determine the existence of an edge. This generalization expanded the model's applicability to real-world systems where connectivity depends on multiple factors beyond simple proximity. Penrose shows, amongst other things, that under some mild assumptions the model is non-trivial.

A further generalization to the RCM is the Marked Random Connection Model (MRCM) which allows for a variety of types. This is used to integrate multiple types of devices into the same model, where each possible pair of devices changes the probability of connection. While various instances of this model have been studied (see Section 3.1) at this level of generality the MRCM was only introduced in [DH22] in 2022. The past decade has seen substantial advances in the understanding the RCM (and its various generalizations). Researchers have developed sharp threshold results, more scaling laws near criticality, and enhanced techniques for analyzing the model's behavior in various regimes (explored in the literature review in Chapter 3).

1.1.1 Some applications

The RCM, MRCM and continuum percolation models more generally have been used to model a large variety of processes since their inception. Edgar Gilbert himself invented the Gilbert disk model while at Bell Labs and had radio stations and epidemiology in mind as possible targets [Gil61].

The RCM is fairly easy to generate while displaying various realistic properties that other models lack, especially with respect to geometric aspects. Thus, it is used broadly as a test for various tests for graph processing (see e.g. [Tos+16]). Properties of the RCM have also been used for statistical estimators [SMj17].

Applications are also found in computer science [DD22] and epidemiology [Bra14]. The model has proven particularly valuable in capturing the inherent randomness in both node placement and connection establishment that characterizes these systems.

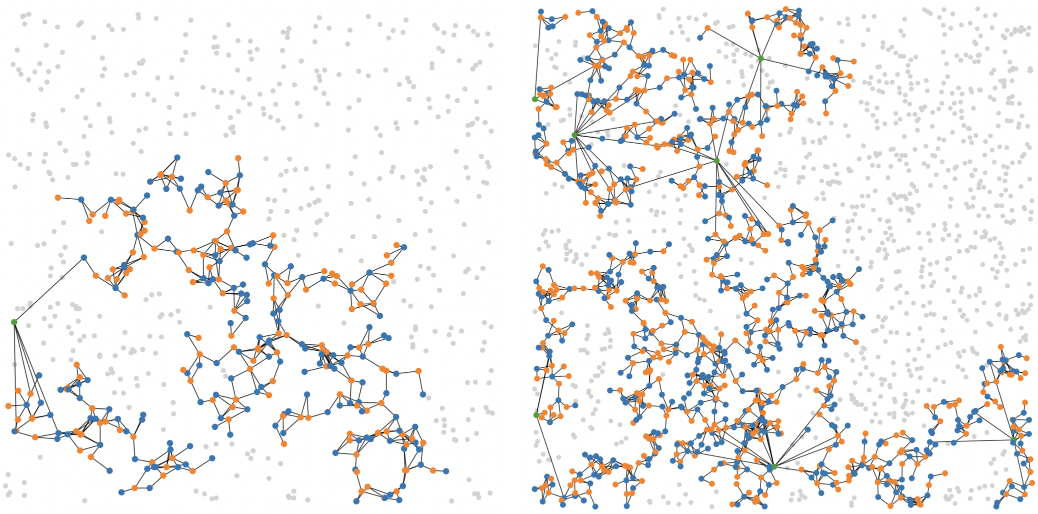

Figure 1-1: Two instances of the same MRCM (with three marks) at two different scales. Largest component in each observation window is highlighted. Within the highlighted component points are colored based on their mark.

The MRCM featured in Figure 1-1 will be used in all future figures as a representation of the MRCM. I will give a short description of it here. It is a modification of AB percolation. In essence, we have two marks (orange and blue), which can only connect to each other but not themselves. And we have a green mark which is very rare, but can connect to both orange and blue over long distances.

1.2 Thesis Aims

The process of generalizing models is an important way to better understand the limits of known results and the effectiveness of our tools. Historically many gains have been made by successful generalization. The aim of this thesis is to continue this tradition by generalizing results and methods for the RCM and MRCM. This goal is guided by the characterization of large components in the sub- and supercritical regimes.

There are two main avenues of achieving these new developments. First, I extend methods which have previously been developed for less general models such as the Poisson Boolean Model or the Random grain model. The other avenue is via 'translating' methods developed for discrete models, such as Bernoulli Bond Percolation or the Random Cluster Model to the MRCM.

The thesis is structured as follows. In Chapter 2 I formally construct the MRCM using *independent edge markings* as in [DH22]. We also consider ways of modifying the MRCM via thinnings and added points. The reader is assumed to have a basic understanding of the Poisson point process, but no knowledge of the RCM or MRCM is required. I also define connected components, the percolation probability and the critical intensity along with other quantities of interest.

In Chapter 3 we review previous results in the field, both giving an overview of the recent history, and taking note of specific results which will be important to our efforts. Further, I will state and prove fundamental theorems which are required to work with the MRCM, most notably the Mecke equation, which converts sums over Poisson points to integrals over the respective density, the Russo formula for interpreting derivatives the Stopping set Lemma which will allow us to consider connected components of the MRCM.

The ultimate goal of Chapters 4 & 5 is to completely characterize the behavior of large components. I will use this goal as motivation to guide the results, but allow some digressions for auxiliary results.

In Chapter 4 we will work with the MRCM in the subcritical phase. The key results will be what well call 'sharpness'. It is required to prove various properties of the correlation length, including its existence. The correlation length will then help us characterize large components by determining the 'correct scale' to view the model at. The intuition for the subcritical phase lies mostly in the fact that large components become exponentially unlikely in their size.

In Chapter 5 we will consider the MRCM in the supercritical phase. In order to characterize large components in this phase we require 'uniqueness' results. By definition of this phase it will be easy to find long paths. The main difficulty in the supercritical phase is ensuring that multiple long paths actually connect to each other with high probability. The uniqueness events allow us to 'glue' such long paths together.

In the final Chapter 6 we will use the tools developed in Chapter 5 to show a classic result from Grimmett and Marstrand [GM90]. This result states that for any supercritical intensity parameter λ we may find a sufficiently thick 'slab', which is only infinite in two dimensions, in which we percolate. We use the Grimmett-Marstrand result to sharpen our tools further, which will allow us to complete the characterization of large components.

Note on Terminology Throughout this thesis, "I" refers to original contributions and perspectives of the author, while "we" is used to include the reader in the mathematical journey and for standard mathematical exposition.

Chapter 2

The Random Connection Model

In this chapter we define the Marked Random Connection Model (MRCM). It is assumed that the reader has some basic understanding of the Poisson point process. I recommend [LP17] as a reference for Poisson point process. We will start by reviewing notation regarding the Poisson point processes, before continuing to with the construction of the MRCM.

2.1 The Poisson Point Process

Let $(\mathbb{X}, \mathcal{X})$ be an arbitrary measurable space. Let $\mathbf{N}(\mathbb{X})$ be the space of measures on $\mathbb{X}$ taking values in $\mathbb{N}$. Let $\mathcal{N}(\mathbb{X})$ be the associated σ-algebra. We will write $ppp(\mathbb{X}, \mu)$ for the law of the Poisson point process on some space $\mathbb{X}$ with intensity measure μ.

Let $d \geq 2$ be an integer. Let $(\mathbb{M}, \mathcal{M}, \rho)$ be a probability space. We call $\mathbb{M}$ the *mark space*. Throughout this thesis we will be working on the space $\mathbb{X} = \mathbb{R}^d \times \mathbb{M}$. Usually, on this space, we will be working with a Poisson point process of uniform intensity $\lambda \in \mathbb{R}_{\geq 0}$ (with respect to $\nu := \mathrm{Leb} \otimes \rho$), in which case will write (by slight abuse of notation) $\eta_\lambda \sim ppp(\mathbb{X}, \lambda)$. We will on occasion drop the subscript to simplify notation.

Now let $\eta \sim ppp(\mathbb{X}, \lambda)$. We will need to modify η by adding additional points. For any point $x \in \mathbb{X}$ we define $\eta^x := \eta + \delta_x$, where δ_x is the Dirac measure. The measure η^x may be interpreted in several ways. First, it may be understood as conditioning on the existence of a point at x: $\mathbb{P}[\eta \in \cdot \mid \eta(\{x\}) = 1] = \mathbb{P}[\eta^x \in \cdot]$. Note that $\eta(\{x\}) = 1$ is an event of probability zero and so the statement is not rigorous as written (see [LP17, Proposition 9.5]). The second way of understanding the Palm distribution is as a shifted Poisson point process. Since the Poisson point process has a uniform distribution it is a priori shift-invariant. However, we may choose a random shift, e.g. choosing the nearest Poisson point to the location x and shifting it to x. The shifted version now necessarily has a Poisson point at x, and thus has a Palm distribution. This property is referred to as 'The Extra Head Problem' (see [LP17, Chapter 10]). Thus, we may view $o \in \eta^o$ as a 'typical' point.

In the same way, for a collection of points $x_1, \dots, x_n \in \mathbb{X}$, we will define $\eta^{x_1,\dots,x_n}$ as $\eta + \sum_{i \leq n} \delta_{x_i}$. Now the interpretation of the 'typical' point is no longer holds, but we will see an interpretation of $\{x_1, \dots, x_n\}$ as a possible instance of a cluster in the next Chapter.

We will also require the factorial measure. For any $\mu \in \mathbf{N}(\mathbb{X})$ of the form of $\mu = \sum_{i \in \mathbb{N}} \delta_{x_i}$ we define the m-th factorial measure

$$\mu^{(m)} := \sum_{(i_1,\dots,i_m) \in \mathbb{N}^m}^{\neq} \delta_{(x_{i_1},\dots,x_{i_m})}.$$

We will be able to safely assume that all the point processes we encounter are of the above form (see [LP17, Corollary 3.7]). The m-factorial measure is the collection of all m-tuples with no repeating points.

2.2 Construction of the MRCM

We are now ready to construct the MRCM. Fix a dimension $d \geq 2$ and mark space $\mathbb{M}$ with an associated probability measure ρ as in the previous section. We will be working on $\mathbb{X} := \mathbb{R}^d \times \mathbb{M}$. This construction follows [DH22] and [Hey+19].

We require a choice of connection function $\psi : \mathbb{R}^d \times \mathbb{M}^2 \to [0, 1]$, which governs the probability that two points in η form an edge based on their marks and their relative position. Given two points $x_a, y_b \in \mathbb{X}$, where $x_a = (\overline{x}, a)$ and $y_b = (\overline{y}, b)$ with $\overline{x}, \overline{y} \in \mathbb{R}^d$ and $a, b \in \mathbb{M}$ we want x_a and y_b to form an edge with probability $\psi(\overline{y} - \overline{x}; a, b)$. To simplify notation we will also write $\psi(\overline{y} - \overline{x}; a, b) = \psi(x_a, y_b)$, where we see $\psi : \mathbb{X}^2 \to [0, 1]$.

Since we are constructing an undirected graph we require ψ to be symmetric in $\mathbb{R}^d$, meaning that for all $a, b \in \mathbb{M}$ and all $\overline{x} \in \mathbb{R}^d$ we have $\psi(\overline{x}; a, b) = \psi(-\overline{x}; b, a)$. The choice of connection function can affect both the microscopic and macroscopic behavior of the MRCM, and we will give various examples in Section 3.1. In this thesis we restrict ourselves to the behavior of the MRCM with connection functions with bounded support.

For any set X we define $X^{[2]}$ as the space of all subsets of cardinality 2. We define the MRCM via a random element ξ of $\mathbf{N}((\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1])$ which we call an *independent edge marking* following a convention by [Hey+19]. We sample it as follows.

Let η be a proper point process on $\mathbb{X}$. We may choose an ordering of the vertices so that $\eta = (x_0, x_1, x_2, \dots)$, where $x_i = (\overline{x}_i, m_i)$. This ordering does not effect the distribution of the resulting MRCM and may thus be chosen freely. Next, we sample a family of random variables $(U_{i,j})_{i,j\in\mathbb{N}}$, where for each $i, j \in \mathbb{N}$ we have $U_{i,j} \sim \mathrm{Unif}([0,1])$ independently of everything else. We then define $\xi[\eta]$ as a point process on $\mathbf{N}((\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1])$ as

$$\xi := \xi[\eta] := \{(\{x_i, x_j\}, U_{i,j}) \mid i < j\}.$$

Throughout most of this thesis we work with the independent edge marking $\xi[\eta]$ where $\eta \sim ppp(\mathbb{X}, \nu)$ and $\nu := \lambda \,\mathrm{Leb} \otimes \rho$ for some $\lambda \geq 0$. In this case, we may write $\xi := \xi[\eta]$. We will use $\mathbb{P}$ for the law of ξ and use $\mathbb{E}$ for the associated expected value. It is important to note that even though η is a Poisson point process, ξ is in general not a Poisson point process. Each point in $(\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1]$ represents an edge, which are strongly correlated.

For convenience we may also want to define the independent edge marking for point processes μ defined on $\mathbb{R}^d$. In this case $\xi[\mu]$ will simply refer to the independent edge marking of the independent ρ-marking of μ (see [LP17, Definition 5.3]).

2.2.1 Adding points

For some $n \in \mathbb{N}$ and distinct points $x_{-1}, \dots, x_{-n} \in \mathbb{R}^d \times \mathbb{M}$ we want to define the MRCM of $\eta^{x_{-1},\dots,x_{-n}}$ in such a way that we may remove (or add) any number of x_i's without changing existing connections. We extend the sequence $(U_{i,j})_{i,j\geq 0}$ to $(U_{i,j})_{i,j\geq -n}$. We define the process $\xi^{x_{-1},\dots,x_{-n}}$ as

$$\xi^{x_{-1},\dots,x_{-n}}[\eta^{x_{-1},\dots,x_{-n}}] := \{(\{x_i, x_j\}, U_{i,j}) \mid i \leq j\}.$$

As before we may not want to explicitly specify the marks. Thus for $\overline{x}_{-1}, \dots, \overline{x}_{-n} \in \mathbb{R}^d$ we define

$$\xi^{\overline{x}_{-1},\dots,\overline{x}_{-n}} := \xi^{(\overline{x}_{-1}, m_{-1}),\dots,(\overline{x}_{-n}, m_{-n})},$$

and sample $m_{-1}, \dots m_{-n} \sim \rho$ independently. For the purpose of compressing notation we may occasionally write

$$\xi^{x_{-1},\dots,x_{-n}} := \xi[\eta^{x_{-1},\dots,x_{-n}}] := \xi^{x_{-1},\dots,x_{-n}}[\eta^{x_{-1},\dots,x_{-n}}].$$

We interpret ξ as a graph in the following manner. The vertices are given by $\eta = (x_0, x_1, x_2, \dots)$. We say two vertices $x_i, x_j \in \eta$ share an edge if $U_{i,j} \leq \psi(\overline{x}_i - \overline{x}_j; m_i, m_j)$ where w.l.o.g. $i < j$. In this case we write $x_i \sim x_j$. Notice that once we have a realization ξ, determining the graph is a completely deterministic operation.

It will be notationally convenient to not always distinguish between points $x \in \mathbb{X}$ and $x \in \mathbb{R}^d$ with full rigor at all times. We shall point out the differences when they appear and matter, and whenever reference is made to a point in $\mathbb{R}^d$ it should be assumed that a mark is sampled for it accordingly. To help with notational efficiency I will on occasion write the mark of a point as a subscript: $x_m = (x, m) \in \mathbb{X}$. It should be noted that for any event A and $x \in \mathbb{R}^d$ it holds that:

$$\mathbb{P}[\xi^x \in A] = \int_{\mathbb{M}} \mathbb{P}[\xi^{x_m} \in A]\rho(\mathrm{d}m).$$

2.2.2 Subsets

We define two variations of ξ, each of which is a subset of the possible edges. Let $\mu, \mu' \subset \mathbb{X}$ be two point processes. We define

$$\xi[\mu, \mu'] := \{(\{x, y\}, u) \in \xi[\mu \cup \mu'] \mid x, y \in \mu \text{ or } x \in \mu, y \in \mu'\} \tag{2.1}$$

to be the independent edge marking which contains all edges with at least one endpoint in μ. Note that the ordering matters. In particular, it does not contain any of the edges which have both end points in μ'.

We also define

$$\xi[\mu; \mu'] := \{(\{x, y\}, u) \in \xi[\mu \cup \mu'] \mid x, y \in \mu \text{ or } x, y \in \mu'\}$$

the set of edges which have both endpoints in μ or both endpoints in μ'. This will be useful notation for considering connected components. Although this definition is symmetric, we will usually interpret the first component μ as random, and the second component μ' as fixed.

2.2.3 Thinnings

An important tool for the MRCM is the ability to precisely partition points based on certain properties, in such a way that we may view each partition independently. We can do this via thinnings.

We will require an additional mark on η for the required randomness. This can be achieved by extending the mark space to $\tilde{\mathbb{M}} := \mathbb{M} \times [0,1]$ with probability measure $\rho \otimes \mathrm{Unif}([0,1])$.

Definition 2.1 (Thinnings)**.** For any function $f : \mathbb{X} \to [0,1]$ we define the f-thinning of η as

$$f_*\eta := \{(x, m, u) \in \eta \mid u \leq f(x, m)\}.$$

We will functions of the form $f : \mathbb{X} \to [0,1]$ *thinning functions.*

For η^1 and η^2 independent copies of η notice that by the superposition principle we can sample $f_*\eta^1$ and $(1-f)_*\eta^2$ independently, and overlay them to recover a copy of η (in distribution). One must be careful when working with thinnings as $\eta \setminus f_*\eta \neq (1-f)_*\eta$, since the two sides rely on different sources of randomness. Equality does hold in distribution.

One important family of thinning functions is defined for all locally finite sets $A \subset \mathbb{X}$ (respectively $A \subset \mathbb{R}^d$) and is the probability of a point $x \in \mathbb{X}$ connecting to A:

$$\psi^A(x) := \mathbb{P}[x \sim A \text{ in } \xi[A \cup \{x\}]] = 1 - \prod_{z \in A}(1 - \psi(x, z)). \tag{2.2}$$

We will also explicitly define a special thinned Poisson point process $\eta_{\langle A \rangle} := \eta \setminus \psi_*^A \eta$. This is equal in distribution to the set $\{x \in \eta \mid x \nsim A \text{ in } \xi[\eta \cup A]\}$. When talking about $\eta_{\langle A \rangle}$ we say that the vertices in $\psi_*^A \eta$ were 'killed' by A. Note that we have $\psi_*^A \eta \cup \eta_{\langle A \rangle} = \eta$ (by definition), where the union is disjoint. Note further that $\eta_{\langle A \rangle} \overset{d}{=} (1 - \psi^A)_* \eta$, but $\eta_{\langle A \rangle} \neq (1 - \psi^A)_* \eta$.

We will differentiate between $f_*\eta^x$ and $f_*\eta \cup \{x\}$, where the former allows for x to be killed, and the latter does not. Further note that if one would want to apply multiple independent thinnings this can be achieved by repeating the above process, including further extending the mark space.

For the purpose of compact notation we will not differentiate between points, sets and vectors in the superscript ot $\psi^{\cdot}$, and it should be assumed that all sets and vectors are 'unpacked'. As an example, for some $x \in \mathbb{X}$, $Y = \{y_1, \ldots, y_n\} \subset \mathbb{X}$, and $\vec{z} \in \mathbb{X}^k$ we write $\psi^{x,Y,\vec{z}} = \psi^{\{x,y_1,\dots,y_n,z_1,\dots,z_k\}}$ for some $n, k \in \mathbb{N}$.

2.3 Quantities of Interest

For the rest of this thesis we will work only with a connection function with bounded support in $\mathbb{R}^d$. We may, without loss of generality, assume that $\operatorname{supp}\psi \subset B_1 \times \mathbb{M}^2$. To see why, consider the scenario where $\operatorname{supp}\psi \subset B_R \times \mathbb{M}^2$ for some $R \in \mathbb{R}_{>0}$. Then the function $x \mapsto \psi(R \cdot x; a, b)$ has support in the unit ball as desired for all $a, b \in \mathbb{M}$. We are required to modify the Poisson point process in tandem to ensure even scaling. We choose $\tilde{\eta} = \{x/R : x \in \eta\}$. This rescaled Poisson point process has intensity $R^d\lambda$.

We will use the notation

$$Z_\psi := \iiint_{\mathbb{R}^d \times \mathbb{M}^2} \psi(y; a, b) \mathrm{d}y \rho^{\otimes 2}(\mathrm{d}(a, b)),$$

where $\rho^{\otimes k}$ refers to the k-fold product with itself for some $k \in \mathbb{N}$. We can interpret λZ_ψ as the expected number of neighbors of a typical point. We also define

$$Z_\psi^\infty := \operatorname*{ess\,sup}_{a \in \mathbb{M}} \iint_{\mathbb{R}^d \times \mathbb{M}} \psi(y; a, b) \mathrm{d}y \rho(\mathrm{d}b). \tag{2.3}$$

It is immediately apparent that $Z_\psi^\infty \geq Z_\psi$. We can similarly interpret λZ_ψ^∞ as the upper bound on the expected number of neighbors given the mark of the origin.

By a slight abuse of notation, for points $x \in \eta$ we will write $x \in A$ for some $A \subset \mathbb{R}^d$ if $x \in A \times \mathbb{M}$. Similarly, we will write $\eta \cap A$ to be the (random) set $\{(\overline{x}, m) \in \eta \mid \overline{x} \in A\}$. On occasion we may write $\xi \cap A$ to mean $\xi[\eta \cap A]$, for the purpose of simplifying notation.

Now that we have defined the MRCM we can define some important objects required to reason about its behavior. First we say that there exists a path from $x \in \eta$ to $y \in \eta$ in a realization $\xi[\eta]$ if we can find a sequence of n distinct points $x = z_0, z_1, z_2, \dots, z_n, z_{n+1} = y$ such that for all $i \in [\![0, n]\!]$ we have $z_i \sim z_{i+1}$. We write $\{x \leftrightarrow y \text{ in } \xi[\eta]\}$ to denote the event "there exists a path from x to y". We will use the convention that if $x = y$ then $x \leftrightarrow y$ holds.

Definition 2.2 (Two-point function)**.** We define the two-point function for $x, y \in \mathbb{R}^d$ or $x, y \in \mathbb{X}$ as

$$\tau_\lambda(x, y) := \tau(x, y) := \mathbb{P}[x \leftrightarrow y \text{ in } \xi[\eta^{x,y}]].$$

We further define the restricted two-point function as

$$\overline{\tau}_\lambda(x,y) := \mathbb{P}[x \leftrightarrow \psi_*^y\eta \text{ in } \xi[\eta^x]], \tag{2.4}$$

i.e. the probability that x reaches a neighbor of y. It can also be interpreted as the probability that x and y connect via at least one other point in η. We can see that $\overline{\tau}$ is symmetric in x and y. It is immediate that $\tau \geq \overline{\tau}_\lambda \geq \tau - \psi$.

Definition 2.3. For a measurable subset $A \subset \mathbb{X}$ we write $\{x \leftrightarrow A \text{ in } \xi[\eta]\}$ if there exists some $y \in \eta \cap A$ such that $x \leftrightarrow y$. Similarly, for $A, B \subset \mathbb{X}$, we write $\{A \leftrightarrow B\}$ in $\xi[\eta]$ if there exist $x \in \eta \cap A$ and $y \in \eta \cap B$ such that $x \leftrightarrow y$ holds. Note that by our convention, if there exists some $x \in \eta \cap A \cap B$ then $A \leftrightarrow B$ holds.

For marks we define a similar connection event. For a measurable subset $M \subseteq \mathbb{M}$ and $x \in \eta$ we write $\{x \leftrightarrow M \text{ in } \xi[\eta]\}$ if there exists some $y \in \eta$ such that x connects to y and y has its mark in M, including x itself.

If we can find an arbitrarily long path of distinct points from x we write $x \leftrightarrow \infty$ in $\xi[\eta]$. Similarly, for some $A \subset \mathbb{X}$, we write $A \leftrightarrow \infty$ if there exists some $x \in \eta \cap A$ such that $x \leftrightarrow \infty$.

Definition 2.4 (Connected Components)**.** We define the connected component of a vertex $x \in \eta$ as follows.

$$\mathcal{C}(x,\xi) := \{y \in \eta \mid x \leftrightarrow y\}.$$

We may also consider the connected component of a region $A \subset \mathbb{R}^d$, in which case we write:

$$\mathcal{C}(A,\xi) := \{y \in \eta \mid A \leftrightarrow y\} = \bigcup_{x \in \eta \cap A} \mathcal{C}(x,\xi).$$

Note that $\mathcal{C}(A,\xi)$ is in general not a connected component. Finally, it will be convenient to define $\mathcal{C}_x := \mathcal{C}(x,\xi^x)$ as the connected component of the added vertex $x \in \mathbb{X}$ (or $x \in \mathbb{R}^d$ in which case the mark is sampled randomly from ρ).

For all $r \in \mathbb{R}_{\geq 0}$ we define the box of radius r to be $\Lambda_r := [-r,r]^d \times \mathbb{M} \subset \mathbb{X}$. Similarly we define $\overline{\Lambda}_r := [-r,r]^d \subset \mathbb{R}^d$. For all $r \in \mathbb{R}_{geq0}$ we define the ball of radius r to be B_r.

The percolation probability is one of the most important quantities across all of statistical physics. Together with the r-percolation probability it helps characterize many important behaviors.

Definition 2.5 (Percolation probability)**.** Let $r \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{M}$. We define the

r-percolation probability as

$$\theta_r^m(\lambda) := \theta_r^m(\lambda, \psi) := \mathbb{P}[o_m \leftrightarrow \Lambda_r^c \text{ in } \xi^o].$$

We define the percolation probability as

$$\theta^m(\lambda) := \theta^m(\lambda, \psi) := \mathbb{P}[o_m \leftrightarrow \infty \text{ in } \xi^o].$$

We write $\theta(\lambda) = \int_{\mathbb{M}} \theta^m(\lambda)\rho(\mathrm{d}m)$, and analogously for $\theta_r(\lambda)$.

Remark 2.6. Note that the choice of reaching the complement of the box Λ_r is somewhat arbitrary. Any other shape such as the ball B_r would work just as well. Indeed the limit to $\theta(\lambda)$ will be the same regardless, as long as your chosen family of shapes is sufficiently nice.

Remark 2.7. It is straightforward to show that $\lim_{r\to\infty} \theta_r(\lambda) = \theta(\lambda)$. The r-percolation probability is monotonically decreasing in r and bounded from below by 0, ensuring the limit exists. The quantity $\theta(\lambda)$ can also be interpreted as the probability that the origin is connected to the unique infinite component, should it exist. Uniqueness of the infinite component is established in [MR96] for the RCM. For the MRCM it was shown in [CL24].

Remark 2.8. By coupling ξ_λ in λ, which is performed in Lemma 4.6, it can be easily shown that $\theta(\lambda)$ is non-strictly increasing in λ.

The percolation probability has two distinct phases in λ. The subcritical phase where $\theta(\lambda) = 0$, and the supercritical phase where $\theta(\lambda) > 0$.

Definition 2.9 (Critical Parameter)**.** The critical parameter λ_c is defined as

$$\lambda_c = \lambda_c(\psi) = \inf\{\lambda \mid \theta(\lambda) > 0\}.$$

A value of λ is said to be *subcritical* if $\lambda < \lambda_c$. We define $\inf \varnothing = \infty$, so if $\psi \equiv 0$, then $\lambda_c = \infty$.

The critical parameter is extraordinarily important. It captures the most important behavior of statistical physics models. This thesis only studies the model below and above the critical threshold, however the behavior at criticality is of great interest, but typically difficult to study.

Remark 2.10. It is not clear that λ_c is non-trivial, meaning that $\lambda_c \in (0, \infty)$. We will cover this in the Literature Review. It is, however, clear that $\theta(\lambda) > 0$ implies the existence of an infinite component. It is less clear that $\lambda < \lambda_c$ implies $\mathbb{E}[|\mathcal{C}_o|] < \infty$. Indeed, we will prove this in Chapter 4. At criticality it is expected that $\theta(\lambda) = 0$ and $\mathbb{E}[|\mathcal{C}_o|] = \infty$.

We can observe similar non-trivial behavior when considering the symmetric simple random walk $(X_i)_{i\geq 0}$ on $\mathbb{Z}$. It is well know that X_i is recurrent, meaning that no matter where it starts it will return to $0 \in \mathbb{Z}$ in finite time with probability 1. However, the expected return time is ∞.

Chapter 3

Literature Review

We start with a brief survey of the various models that are covered by the Marked Random Connection Model (MRCM). This will hopefully aid with the intuition for later sections and chapters.

3.1 Examples of the MRCM

The MRCM is a very general model than can capture a large variety of different behaviors. I will present some instances of the MRCM which have been are of interest. These are useful to keep in mind as we proceed.

We will divide this exposition of models into *hard* and *soft*. We call a model hard if connections are purely determined by the relative location of points and their marks, but no other randomness. If extra randomness is required to determine the edges, we call the model soft.

3.1.1 Hard Models

First, let us mention the simplest model, which is the Poisson Boolean model of constant radius, also referred to as the Gilbert graph. We let $|\cdot|_2$ denote the standard L^2-norm on $\mathbb{R}^d$. It can be understood to have the connection function $\psi(x) = \mathbf{1}\{|x|_2 \leq 1\}$. This model is by far the best understood.

The first generalization of the Gilbert graph was allowing the radii to vary. In this case we choose $\mathbb{M} = \mathbb{R}_{\geq 0}$ and along with some distribution ρ. We write marks as subscripts. The connection function is then $\psi(x_r, y_s) = \mathbf{1}\{|x - y|_2 \leq r + s\}$. For this model to be non-trivial it is required that $E[R^d] < \infty$, where $R \sim \rho$, although it is common to consider stronger moment assumptions, or even bounded radii. See [MR96].

Grain models are the most general of the hard models. The mark space is the space of all compact sets containing the origin. If K, L are compact subsets of $\mathbb{R}^d$, then we write $\psi(x_K, y_L) = \mathbf{1}\{x + K \cap y + L \neq \varnothing\}$. See [Zie16] for an exploration of this model.

AB percolation is a modification of any of the above models. In AB percolation we consider the mark space $\tilde{\mathbb{M}} = \{A, B\}$ (in addition to any other marks) and only allow connections between points of different marks. See [IY12] [Pen14]. Of course, AB percolation can also be performed on soft models.

3.1.2 Soft Models

The tools required for tackling the soft models have only been developed more recently. As mentioned before, the first of these models to be studied was the RCM in [Pen91].

One instance of the RCM of interest is first sampling a hard model, and then performing Bernoulli bond percolation on the resulting random graph, this was studied by [Pen22] and [Lic+23]. This can also be interpreted as using the connection function $\psi(x, y) = p\mathbf{1}\{|y - x|_2 \leq 1\}$, for some parameter $p \in (0, 1)$. Note that this principle can be applied to any model. Moreover, we can find a critical value p_c^{bond} analogous to our critical λ_c. It was shown in [FPR11] that this critical value is strictly smaller than the critical value for site percolation performed on the RCM, i.e. $p_c^{\text{site}} > p_c^{\text{bond}}$. This is consistent with previous results on graphs of bounded degree [GS98].

One particular model of interest is the weight-dependent RCM (introduced in [Gra+22]) which is usually written with mark space $\mathbb{M} = [0, 1]$. This model depends on a non-increasing integrable profile function $p : \mathbb{R}_{\geq 0} \to [0, 1]$ and a *kernel* $g : (0, 1) \times (0, 1) \to \mathbb{R}_{\geq 0}$. Then $\psi(x_t, y_s) = p(g(t, s)|x - y|^d)$. This model is of particular interest when studying long range models, i.e. models where the connection distance has a polynomial tail, and hence long range connections become important to the dynamics. See also [Gra+19].

3.2 Overview of previous results

We start by going through fundamental results necessary for understanding any statistical physics model. We start with uniqueness and sharpness. We will then continue to recent work which is more directly relevant to this thesis. Finally, we will state preliminary lemmas that are required throughout the rest of this thesis.

This thesis builds on recent work in both discrete and continuum percolation. We first go through historical advances in continuum models. Then, we look at some results specific to large components. We then look at some recent results for the MRCM. Finally, we cover the results in discrete models which are of importance to us.

3.2.1 Historical results for Continuum Models

As mentioned in Chapter 1, Penrose introduced the RCM [Pen91] in 1991. In this first paper Penrose shows that the critical intensity in non-trivial. Furthermore, and importantly for our purposes, he derives a formula for the probability that an added point connects to a cluster of cardinality k. I will generalize this to the MRCM and present a new proof in Chapter 4.

One foundational text for continuum models is Meester and Roy's *Continuum Percolation* [MR96]. Meester and Roy largely work with the Poisson Boolean model (of varying radius), proving standard results. For our purposes we are interested in their work on the RCM. They show the agreement of two separate definitions of criticality, one given by the percolation probability $\theta(\lambda)$ (aligning with our Definition 2.9) and the other given by the expected size of the component at the origin $\mathbb{E}[|\mathcal{C}_o|]$. They also show the uniqueness of the infinite component when it exists. Their proof makes use of the 'trifurcation' argument introduced by Burton and Keane in [BK89].

Uniqueness of the infinite component for the RCM was first shown by Meester and Roy (see [MR96]) using the Burton-Keane approach [BK89], where the key idea is the use of 'trifurcation-points'. The approach of 'trifurcation-points' also works in the continuum. For the MRCM uniqueness was shown in [CL24], which adapts ideas from [AKN87], in particular deletion stability, to prove uniqueness.

3.2.2 Large components

Another important text is Penrose's *Random Geometric Graphs* [Pen03]. Penrose shows for the Poisson Boolean model of constant radius, i.e. $\psi(x) = \mathbf{1}\{|x| \leq 1\}$, that the largest component in a box Λ_t is of the order of the $\log(t)$ in the subcritical case. In the supercritical case the largest component in Λ_t consists of a positive fraction $\theta(\lambda)$ of all points in the box. We will prove these in the more general MRCM.

In the case where ψ is radially symmetric and decreasing Penrose proved in [Pen16] that full connectivity (in a box) is governed by the probability of having isolated vertices in the limit where the number of points goes to infinity, but the radius of the connection function goes to zero. Furthermore, the number of isolated vertices can itself be approximated by a Poisson random variable.

Penrose showed in [Pen22] that the supercritical ($\lambda > \lambda_c$) largest component of the as a fraction of all points converges to $\theta(\lambda)$ in probability for $d = 2$ for ψ decreasing with bounded support. In the case where $\psi(x) = p\mathbf{1}\{|x| \leq 1\}$ for some $p \in (0, 1]$ Lichev, Lodewijks, Mitsche and Schapira show in [Lic+23] that the largest component in the supercritical SRGG converges to $\theta(\lambda)$ almost surely as a fraction of all points. They further demonstrate that the critical parameters $\lambda_c(p)$ and $p_c(\lambda)$ are inverses of the other.

3.2.3 Recent advances for the (M)RCM

One open question is if the infinite cluster exists at criticality. This property is equivalent to $\theta(\lambda)$ being continuous at λ_c[1]. For discrete models such as Bernoulli bond percolation on $\mathbb{Z}^2$ this was shown by Harris in 1960 [Har60], when taken together with a later result by Kesten in 1980 [Kes80]. For a general survey of planar percolation see [Gri99]. In high dimensions, namely $d \geq 19$ the result was shown by Hara and Slade in 1994 using the Lace Expansion [HS94] (in particular they show "mean-field behavior").

Last and Ziesche show that the two-point function satisfies the Ornstein-Zernike equation [LZ17]. Later, Heydenreich van der Hofstad, Last and Matzke develop the lace expansion for the two point function [Hey+19], which makes this relationship explicit, this allows them to derive the triangle-condition (in sufficiently high dimensions). This paper is important for our purposes due to the Stopping Set lemma which we will state and prove later.

In general all models are least well understood at or near the critical value. One way we try to understand the behavior of models near the critical value is with critical exponents, each of which describe a certain aspect of the model behavior. It is conjectured that the critical exponents are universal, meaning that they are independent of the exact details / local rules of the model.

Dickson and Heydenreich prove in [DH22], under some mild technical assumptions, that the 'triangle condition' holds in sufficiently high dimensions. Using the triangle condition [CD24] show the existence (and calculate the values of) certain critical exponents. For the RCM [DH24] develop an expansion of the critical value λ_c in the limit as $d \to \infty$. Relevant to this paper, [CD24] show the 'mean-field lower bound' half of sharpness.

[1] As we will see it is easy to prove right-continuity, the difficulty comes in showing θ is left-continuous.

3.2.4 Recent Advances in Discrete Models

More recently lace expansion result has since been generalized to graphs beyond $\mathbb{Z}^d$ such as [Ben+99] for Cayley graphs of non-amenable groups and [Hey+19] for the Random Connection Model (in high dimensions). For low dimensions it was shown by Duminil-Copin, Sidoravicius and Tassion in [DST16] that there is no infinite cluster at criticality for 'slabs' of the form $\mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$.

The property sharpness originally referred to the coincidence of different definitions of of criticality, one defined via the probability of an infinite path (as we have done in Definition 2.9), and the other via the expected cluster size. This fact was independently discovered by Aizenman and Barsky [AB87] and Menshikov [Men86]. Menshikov and Sidorenko later showed the same result for Poisson models [MS88].

An important modern result on sharpness is [DT16] by Duminil-Copin and Tassion, which greatly simplifies (and generalizes) the proof. In particular, [DT16] shows 'exponential decay' of the t-percolation probability (in t) in the subcritical case, and a mean-field lower bound for the percolation probability in the supercritical case. For the rest of this thesis 'sharpness' will refer to this pair of bounds on the percolation probability.

This proof was adopted by Sebastian Ziesche to show sharpness for the germ model in [Zie16]. In this thesis I generalize this proof to the more general Marked Random Connection Model using methods from [Hey+19]. These new results will allow us to generalize results on large components from [Pen03], [Pen22] and [Lic+23] to the RCM and to dimensions $d \geq 2$.

In [EST24] Easo, Severo and Tassion invert the classic Peirels argument to show $p_c < 1$ is equivalent to at most exponential growth in the number of cutsets (sets of edges separating the origin from infinity) in the size of the cutest.

In Chapter 5 we will state and prove that large components in the supercritical case make up a $\theta(\lambda)$ proportion of all points. The key tools required to work in the supercritical regime are uniqueness statements which allow us to ensure that various crossings of annuli are indeed the same component. We will adapt a lot of the work of [CMT23]. A fundamental result is an upper bound on the two-arm event. The two-arm event occurs when there are two disjoint paths from the origin to the region outside a ball centered at the origin.

3.3 Fundamental Tools

In the following subsections we will be stating fundamental results required to work with the MRCM. These are the Mecke formula, Russo's formula and the Stopping set Lemma. I will also provide a proof of the stopping set lemma, since we will use not only the statement itself, but aspects of its proof in later sections.

3.3.1 Mecke formula

For this and all further sections we shall assume that our Poisson point process η has density $\lambda \,\mathrm{Leb} \otimes \rho$ in $\mathbb{X}$, for some constant $\lambda \geq 0$. In particular, η is uniform on Euclidean space.

A key tool which makes working with the MRCM tractable is the Mecke formula. In words, the Mecke formula allows us to convert the expectation of a sum over Poisson points into an integral with respect to a single point. We will first write out the generalized multivariate version where we sum over m-tuples, and then we will separately consider the univariate version, which we will be using most of the time.

Let us write $\nu := \lambda \,\mathrm{Leb} \otimes \rho$.

Theorem 3.1 (Multivariate Mecke)**.** *Let $\xi_\lambda = \xi[\eta]$ be a MRCM. Let $k \geq 1$. Given a nice function $f : \mathbf{N}((\mathbb{X} \times \mathbb{M})^{[2]} \times [0,1]) \times \mathbb{X}^k \to \mathbb{R}_{\geq 0}$. Then*

$$\mathbb{E}\left[\sum_{\vec{x} \in \eta^{(k)}} f(\xi, \vec{x})\right] = \int_{\mathbb{X}^k} \mathbb{E}[f(\xi^{x_1,\dots,x_k}, \vec{x})] \nu^{\otimes k}(\mathrm{d}\vec{x}),$$

where $\nu^{\otimes k}$ is the k-times product of the measure ν, and $\vec{x} = (x_1, \dots, x_k)$.

Remark 3.2. Depending on our needs we may not want to explicitly integrate over possible marks in the expected value. By abuse of notation we will not, in general, differentiate between various $\mathbb{E}$.

$$\int_{\mathbb{X}^k} \mathbb{E}[f(\xi^{x_1,\dots,x_k}, \vec{x})] \mathrm{d}\nu^{\otimes k}(\mathrm{d}\vec{x}) = \lambda^k \int_{\mathbb{R}^{dk}} \mathbb{E}[f(\xi^{\overline{x}_1,\dots,\overline{x}_k}, \overline{x})] \mathrm{d}\overline{x}.$$

On the LHS we explicitly integrate over the marks while on the RHS the integral over the marks is carried out by the expected value.

In the special case where $k = 1$ we recover the univariate Mecke formula. As we will use this form in the majority of cases it is worth stating separately.

Theorem 3.3 (Univariate Mecke)**.** *Let* ξ_λ *be a MRCM. Given a nice function* f : $\mathbf{N}((\mathbb{X} \times \mathbb{M})^{[2]} \times [0,1]) \times \mathbb{X} \to \mathbb{R}_{\geq 0}$*. Then*

$$\mathbb{E}\left[\sum_{\vec{x} \in \eta} f(\xi, x)\right] = \lambda \int_{\mathbb{X}} \mathbb{E}[f(\xi^x, x)]\nu(\mathrm{d}x).$$

If the sum is over a Palm process, the extra points are added on the RHS. For the univariate Mecke equation this results in:

$$\mathbb{E}\left[\sum_{\vec{x} \in \eta^z} f(\xi, x)\right] = \mathbb{E}[f(\xi^z, z)] + \int_{\mathbb{X}} \mathbb{E}[f(\xi^x, x)]\nu(\mathrm{d}x).$$

Proof of Multivariate Mecke. By [LP17, Theorem 4.4], we know the Multivariate Mecke equation holds for the (marked) Poisson point process. We write $\mathbb{E}_\eta$ to refer to the expected value with respect to the point process without the extra randomness from the MRCM. Similarly, we write $\mathbb{E}_{U_{ij}}$ to refer to the expected value with respect to the MRCM. To extend to the MRCM we have to incorporate the extra randomness coming from ξ. Then

$$\begin{aligned}
\mathbb{E}\left[\sum_{\vec{x} \in \eta^{(k)}} f(\xi, \vec{x})\right] &= \mathbb{E}_{(U_{ij})}\left[\mathbb{E}_\eta\left[\sum_{\vec{x} \in \eta^{(k)}} f(\xi, \vec{x})\right]\right] \\
\text{(Mecke for ppp)} \quad &= \mathbb{E}_{(U_{ij})}\left[\lambda^k \int_{\mathbb{X}^k} \mathbb{E}_\eta\left[f(\xi^{\vec{x}}, \vec{x})\right] \nu^{\otimes k}(\mathrm{d}\vec{x})\right] \\
\text{(Fubini)} \quad &= \lambda^k \int_{\mathbb{X}^k} \mathbb{E}[f(\xi^{\vec{x}}, \vec{x})]\nu^{\otimes k}(\mathrm{d}\vec{x}).
\end{aligned}$$

One needs to be careful when separating the expected value $\mathbb{E}$ into $\mathbb{E}_{U_{ij}}\mathbb{E}_\eta$. The labels i need to be assigned to the Poisson points consistently, e.g. by the distance from the origin. ■

Going forward we will just write ‘dx’ when integrating over $\mathbb{X}$ for brevity.

3.3.2 Margulis-Russo Formula

This formula is standard on lattice models (see e.g. [Gri99]), and known for the Poisson point process [LP17, Chapter 19]. It was expanded to the RCM in [LZ17, Theorem 3.2]. Their proof can be adapted to the MRCM *mutatis mutandis.*

Let $f : \mathbf{N}(\mathbb{X}^{[2]} \times [0,1]) \to \mathbb{R}$ be a function. We say the function f lives on a Borel set $\Lambda \subset \mathbb{R}^d$ if $f(\xi[\eta \cap \Lambda]) = f(\xi[\eta])$ almost surely.

Theorem 3.4 (Margulis-Russo Formula)**.** *Assume f lives on some bounded Λ and there exists some $\lambda_0 > 0$ such that $\mathbb{E}[f(\xi_{\lambda_0})] < \infty$. For every $\lambda \leq \lambda_0$ we have*

$$\frac{\partial}{\partial \lambda}\mathbb{E}[f(\xi_\lambda)] = \int_\Lambda \mathbb{E}[f(\xi_\lambda^x) - f(\xi_\lambda)]\mathrm{d}x.$$

The Margulis-Russo formula is a fundamental result, which will allow us to construct a differential inequality, and more generally unlocks the use of calculus for studying the MRCM.

3.3.3 Stopping Set Lemma

The Stopping Set Lemma (introduced in [Hey+19]) is a central tool that we will use liberally throughout this thesis. It will allow us to disintegrate along components of the MRCM. Specifically, it allows us to work with $\mathbb{P}[\ \cdot \mid \mathcal{C} = C]$ for some random component $\mathcal{C}$ and an admissible deterministic point set C.[2]

Definition 3.5 (Admissible Set)**.** We call a finite set $C \subset \mathbb{X}$ *admissible* if the probability that C is connected in $\xi[C]$ is strictly positive. Let G_C be the set of all possible graphs on the vertices C such that the graph is connected. Then

$$g(C) := \mathbb{P}[C \text{ is connected in } \xi[C]] = \sum_{G \in G_C} \prod_{xy \in E(G)} \psi(x,y) \prod_{xy \notin E(G)} (1 - \psi(x,y)), \quad (3.1)$$

where xy represents an edge with endpoints x and y.

The Stopping Set Lemma allows us to work with the modified point process $\eta_{\langle C \rangle}$ (defined in Section 2.2) instead of $\xi^x[\eta^x \setminus \mathcal{C}_x]$. Note that $\eta^x \setminus \mathcal{C}_x$ is not a Poisson point process in general.

Lemma 3.6 (Stopping Set lemma)**.** *Let $x \in \mathbb{R}^d$. Then*

$$\mathbb{P}[\xi^x[\eta^x \setminus \mathcal{C}_x] \in \cdot \mid \mathcal{C}_x = C] = \mathbb{P}[\xi[\eta_{\langle C \rangle}] \in \cdot] \qquad \text{for } \mathbb{P}[\mathcal{C}_x \in \cdot\ ] \text{ a.e. } C.$$

Let us first note that $\eta^x \setminus \mathcal{C}_x$ is not a Poisson point process.

Due to the centrality of this lemma and for completeness we will give the proof of

[2]Note that $\mathcal{C} = C$ may be an event of measure zero. However, conditioning on this event can be made rigorous using the disintegration theorem. See for instance [Pac78].

[Hey+19, Lemma 3.3] here. In particular, this proof includes details which will be of importance in Chapter 5. The authors of [Hey+19] note that this proof is similar to [MPS97, Proposition 2].

Proof. We write $\xi^x[\mathcal{C}_x]$ to refer to $\mathcal{C}_x$ with the edges. Notice that $\xi^x[\mathcal{C}_x]$ can be interpreted as a rooted tree with x as the root. Let $\eta_0 = \{x\}$. We now iteratively construct $\xi^x[\mathcal{C}_x]$. For $n \in \mathbb{N}$ let η_n be the set of points in $\mathcal{C}_x$ with graph distance at most n from the root x. Let $C_0 = \eta_0 = \{x\}$. We claim the following holds

$$\begin{aligned}&\mathbb{E}[f(\xi^x[\eta \setminus \eta_n], \eta_0, \dots, \eta_n)]\\ &= \int \mathbb{E}[f(\xi[\eta_{\langle A_{n-1}\rangle}], A_0, \dots, A_n)]\mathbb{P}[(\eta_0, \dots, \eta_n) \in \mathrm{d}(A_0, \dots, A_n)],\end{aligned} \tag{3.2}$$

for all measurable f with suitable domain.

To continue we consider a modification of the MRCM ξ. Let $\mu \subset \mathbb{X}$ be a point processes. Let $A \subset \mathbb{X}$ be a locally finite set. Following the conventions from Section 2.2 write $\xi[\mu, A]$ for the set of edges with at least one endpoint in μ (see (2.1)). We assert that for $n \in \mathbb{N}$:

$$\begin{aligned}&\mathbb{E}[f(\xi^x[\eta \setminus \eta_n, \eta_n \setminus \eta_{n-1}], \eta_0, \dots, \eta_n)]\\ &= \int \mathbb{E}[f(\xi[\eta_{\langle A_{n-1}\rangle}, A_n \setminus A_{n-1}], A_0, \dots, A_n)]\mathbb{P}[(\eta_1, \dots, \eta_n) \in \mathrm{d}(A_1, \dots, A_n)],\end{aligned} \tag{3.3}$$

for all non-negative measurable f with suitable domain. Note that this is a stronger assertion than (3.2), because $\xi[\eta_{\langle A_{n-1}\rangle}] \subseteq \xi[\eta_{\langle A_{n-1}\rangle}, A_n \setminus A_{n-1}]$.

This is equivalent to stating that $\xi^x[\eta \setminus \eta_n, \eta_n \setminus \eta_{n-1}]$ is equal to $\xi[\eta_{\langle A_{n-1}\rangle}, A_n \setminus A_{n-1}]$ in distribution given $(\eta_0, \dots, \eta_n) = (A_0, \dots, A_n)$.

We prove (3.3) as follows. Let us define the following edge marking. Let $h : \mathbb{R}^d \to [0, \infty)$ be measurable and let μ and μ' be two independent Poisson point process with their intensity given by h. Let $A \subset \mathbb{R}^d$ be a locally finite set. We define the following independent edge marking $\tilde{\xi} := \xi[\mu \cup A]$. Let μ^A be the set of points in μ directly connecting to A in $\tilde{\xi}$. Observe that for each $v \in \mu$ the event that $v \sim A$ is independent of all other connections in μ. In particular, $\mu^A \overset{d}{=} \psi_*^A \mu'$ (we remind the reader of Definition 2.1). Moreover,

$$\left(\xi[\mu \setminus \mu^A, \mu^A], \mu^A\right) \overset{d}{=} \left(\xi[\mu'_{\langle A\rangle}, \psi_*^A \mu'], \psi_*^A \mu'\right). \tag{3.4}$$

This statement follows from marking and thinning theorems of Poisson point processes

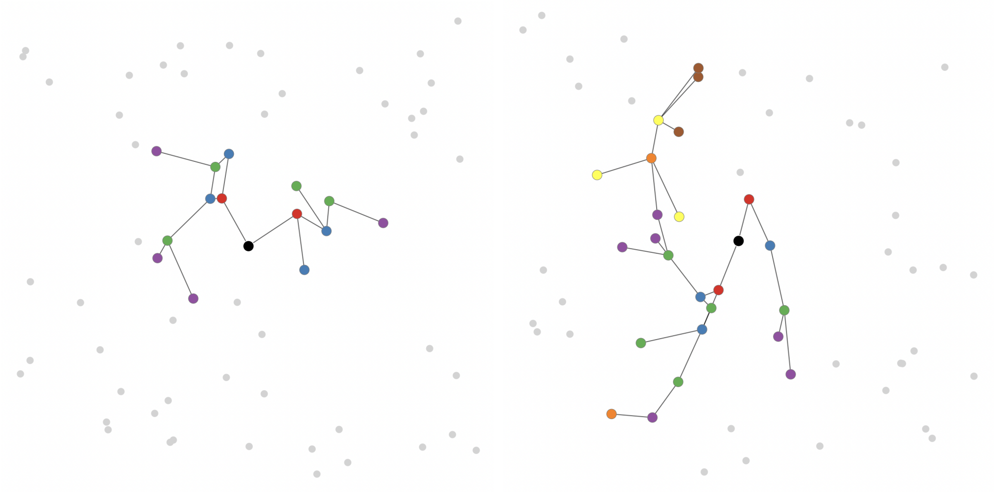

Figure 3-1: Two demonstrations of the iterative construction of a component. Vertices are colored based on their graph distance to the origin (in black).

(see [LP17, Theorems 5.1 & 5.6]). It can also be seen directly by noticing that instead of using the randomness from the $U_{i,j}$'s from $\xi[\cdot]$ we are instead using the randomness from the thinning. Furthermore, $\mu'_{\langle A\rangle}$ and $\psi_*^A\mu'$ are independent.

We now apply (3.4) iteratively starting with $A = A_0 = \{x\}$ and $\mu = \eta$. We construct A_i iteratively. First, let A_1 be the union of $\{x\}$ and all neighbors of $A_0 = \{x\}$. By construction, it holds that $\psi_*^x\eta \stackrel{d}{=} A_1 \setminus A_0$. This gives (3.3) for $n = 1$:

$$\mathbb{E}[f(\xi^x[\eta \setminus \eta_1, \eta_1 \setminus \eta_0], \eta_0, \eta_1)] = \int \mathbb{E}[f(\xi[\eta_{\langle x\rangle}, A_1 \setminus \{x\}], \{x\}, A_1)]\mathbb{P}[\eta_1 \in \mathrm{d}A_1].$$

Note that $\eta_1 \setminus \eta_0 \stackrel{d}{=} \psi_*^{\{x\}}\eta$. Now suppose (3.3) is true for n and let $A_1, \ldots, A_n \subset \mathbb{X}$ be admissible locally finite sets. We apply (3.4) with $\mu = \eta_{\langle A_{n-1}\rangle}$ and $A = A_n \setminus A_{n-1}$, conditional on $(\eta_1, \ldots, \eta_n) = (A_1, \ldots, A_n)$.

To see this, first note that the points directly connecting to $A_n \setminus A_{n-1}$, but not to any previous points, are exactly $\eta_{n+1} \setminus \eta_n$. In symbols, we find that $\eta_{\langle A_{n-1}\rangle}^{A_n\setminus A_{n-1}} = \eta_{n+1} \setminus \eta_n$ given $(\eta_1, \ldots, \eta_n) = (A_1, \ldots, A_n)$, by definition.

Given $(\eta_1, \ldots, \eta_n) = (A_1, \ldots, A_n)$ we find

$$\xi[\eta \setminus \eta_{n+1}, \eta_{n+1} \setminus \eta_n] \stackrel{d}{=} \xi[(\eta_{\langle A_{n-1}\rangle})_{\langle A_n\setminus A_{n-1}\rangle}, \psi_*^A\eta_{\langle A_{n-1}\rangle}]. \tag{3.5}$$

We can see that $(\eta_{\langle A_{n-1}\rangle})_{\langle A_n \setminus A_{n-1}\rangle} \stackrel{d}{=} \eta_{\langle A_n \rangle}$. This follows explicitly from:

$$(1-\psi^{A_{n-1}})(1-\psi^{A_n \setminus A_{n-1}}) = \left(\prod_{z\in A_{n-1}} 1-\psi^z\right)\left(\prod_{z\in A_n\setminus A_{n-1}} 1-\psi^z\right) = 1-\psi^{A_n}.$$

Similarly, we find that, $\psi_*^{A_n\setminus A_{n-1}}\eta_{\langle A_{n-1}\rangle} = A_{n+1}\setminus A_n$. And so (3.3) follows from (3.5).

Furthermore, this same argument shows that, conditionally on $(\eta_k)_{k\le n}$, $\eta \setminus \eta_{n+1}$ and $\eta_{n+1}\setminus \eta_n$ are independent Poisson point processes with intensity functions $\lambda(1-\psi^{\eta_n})$ and $\lambda\psi^{\eta_n}(1-\psi^{\eta_{n-1}})$, respectively. The intuition is that a Poisson point *not* connected to η_n can not be in η_{n+1}. If a Poisson point *is* connected to η_n it has to additionally not be connected to η_{n-1}, as those points have already been sampled.

This procedure allows us to iteratively sample components $\mathcal{C}_x$ by starting with $\eta_0 = \{x\}$ and $\eta_{-1} = \varnothing$, and sampling $\eta_{n+1}\setminus\eta_n$ as an independent Poisson point process with intensity $\lambda\psi^{\eta_n}(1-\psi^{\eta_{n-1}})$.[3]

By induction, integrability of ψ and

$$\psi^{\eta_n\setminus\eta_{n-1}}(\cdot) \le \sum_{w\in\eta_n\setminus\eta_{n-1}} \psi(\cdot, w),$$

it follows that all η_n are finite almost surely. Equation (3.2) shows in particular that

$$\mathbb{E}[f(\xi^x[\eta\setminus\eta_n],\eta_n)] = \int \mathbb{E}[f(\xi[\eta_{\langle \mathcal{C}_{n-1}\rangle}],\mathcal{C}_n)]\mathbb{P}[\xi^x[\mathcal{C}_x]\in \mathrm{d}C], \tag{3.6}$$

where $\mathcal{C}_n$ refers to all vertices with graph distance at most n from x in $\mathcal{C}_x$. Let $\eta_\infty := \cup_n \eta_n$ denote the vertex set $\mathcal{C}_x$. For any bounded Borel set, we have that $|\eta_\infty\cap B| = |\eta_n \cap B|$ for sufficiently large n. Note that $\xi^x[\eta\setminus\eta_n] \searrow \xi^x[\eta\setminus\eta_\infty]$ as $n\to\infty$. Therefore, if f is a bounded function depending only on the values of $\xi^x[\eta\setminus\eta_n]$ and η_n on some bounded and measurable set, then the left-hand side of (3.6) will converge to $\mathbb{E}[f(\xi^x[\eta\setminus\eta_\infty],\eta_\infty)]$ as $n\to\infty$.

Similarly, the integrand of the right-hand side converges. In particular,

$$\mathbb{E}[f(\xi^x[\eta\setminus\eta_\infty],\eta_\infty)] = \int \mathbb{E}[f(\xi[\eta_{\langle C\rangle}],C)]\mathbb{P}[\eta_\infty\in \mathrm{d}C],$$

for non-negative f as described in the previous paragraph. This can then be extended to general f by a monotone class argument. ■

[3] It follows from our definition that $\psi^\varnothing = 0$.

Chapter 4

The Subcritical Regime

The goal of this chapter is to prove statements about large components. The subcritical regime in many ways easier to understand than the supercritical regime. The key fact is that long connections become exponentially unlikely. To do so we will consider the largest component in an observation window Λ_t.

4.1 Assumptions and Results

We borrow the following language from [CD24]. We start with the following assumption. We first define, for $a, b \in \mathbb{M}$,

$$D(a,b) := \int_{\mathbb{R}^d} \psi(x; a, b) \mathrm{d}x,$$

and for $k \geq 1$

$$D^{(k)}(a,b) := \int_{\mathbb{M}^{k-1}} \prod_{j=1}^{k} D(c_{j-1}, c_j) \rho^{\otimes k-1}(\mathrm{d}c_{[\![1,k-1]\!]}),$$

where $c_0 = a$ and $c_k = b$. Let $B \subset \mathbb{M}$ be measurable. We know by the Mecke equation that $\lambda \int_B D(a,b)\rho(\mathrm{d}b)$ is the expected number of direct connections made from (o, a) to some point with a mark in B. Similarly, $\lambda^k \int_B D^{(k)}(a,b)\rho(\mathrm{d}b)$ is the expected number of paths of length k that start at (o, a) and end with some mark in B. We will require the following assumption:

$$\operatorname*{ess\,sup}_{a \in \mathbb{M}} \sup_{k \geq 1} \operatorname*{ess\,inf}_{b \in \mathbb{M}} D^{(k)}(a,b) > 0. \tag{\textbf{A1}}$$

In words (**A1**) says that there exists some mark that connects to every other mark in at most k steps for some k. This immediately implies that almost every mark can connect to almost every other mark in at most $2k$ steps. In particular, it ensures that for ρ-almost all marks m and any measurable subset $M \subset \mathbb{M}$ such that $\rho(M) > 0$ we have that $\mathbb{P}[o_m \leftrightarrow M] > 0$ (in the sense of Definition 2.3).

Assumption (**A1**) is not required for all results and in particular the main theorem would still hold without it. The assumption ensures that our model can not be trivially decomposed into two independent MRCMs. It will be explicitly mentioned whenever (**A1**) is required.

It is worth remarking that (**A1**) does exclude some models which might be of interest. An example could be given by $\psi(x; a, b) = \min(a, b)f(x)$, where the mark space $\mathbb{M} = [0, 1]$, ρ is given by uniform distribution and f is a symmetric function of bounded support.

Definition 4.1 (Largest Component)**.** Let $B \subset \mathbb{X}$. The size of the largest component in $\xi_\lambda \cap B$, as measured by the number of vertices is denoted as $L_1(B, \xi_\lambda)$. When ξ are clear from the context, we will simply write $L_1(B)$. In case of a tie the reader may choose any rule to break ties such as by lexicographic ordering.

To be able to state the theorem we need to define the *inverse correlation length.* The correlation length is a general concept throughout statistical physics and can be thought of as the scale required to observe the effects of subcriticality (respectively supercriticality). In our case it means that your observation window needs to be at least at the scale of the correlation length to observe the exponential decay. This notion will be made more rigorous in the proof of the main theorem of this chapter.

Definition 4.2 (Inverse Correlation length)**.** Let $m \in \mathbb{M}$. The m-inverse correlation length for $\lambda \in \mathbb{R}_{>0}$ is defined as

$$\zeta^m(\lambda) := \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}[|\mathcal{C}_{o_m}| = n \text{ in } \xi_\lambda^{o_m}] \tag{\textbf{C1}}$$

$$= \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}[n \leq |\mathcal{C}_{o_m}| < \infty \text{ in } \xi_\lambda^{o_m}]. \tag{\textbf{C2}}$$

It is not immediately obvious that the limit (**C1**) exists or is equal to (**C2**). We also define the following variations

$$\zeta(\lambda) := \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}[|\mathcal{C}_o| = n \text{ in } \xi_\lambda^o]. \tag{C3}$$

$$\zeta^{\min}(\lambda) := \operatorname*{ess\,inf}_{m\in\mathbb{M}} \zeta^m(\lambda). \tag{C4}$$

We remark that $\mathbb{P}[|C_o| = n \text{ in } \xi_\lambda^o] = \int_{\mathbb{M}} \mathbb{P}[|C_{o_m}| = n \text{ in } \xi_\lambda^{o_m}]\rho(\mathrm{d}m)$.

We will prove that $\zeta(\lambda) = \zeta^{\min}(\lambda)$. Assuming (**A1**), we will further prove that for ρ-almost-all $m \in \mathbb{M}$ is holds that $\zeta^m(\lambda) = \zeta(\lambda)$. Note that if (**A1**) does not hold it need not be true. The simplest counterexample is a two mark system where the marks have no interaction.

Lemma 4.3 (ζ is well defined)**.** *The limit* (**C1**) *exists and is equal to* (**C2**)*. The limit* (**C3**) *exists and is equal to* (**C4**)*. Furthermore, the inverse correlation length is positive, decreasing and continuous for all $\lambda < \lambda_c$. As $\lambda \to 0$ we have $\zeta^m(\lambda) \to \infty$.*

Assuming (**A1**) *it further holds that for almost all $m \in \mathbb{M}$: $\zeta^m(\lambda)$ is equal to $\zeta(\lambda)$ for all $\lambda < \lambda_c$.*

To prove Lemma 4.3 we will require sharpness, and more generally the results of Section 4.3. The (inverse) correlation length is a key tool in the study of percolation theory. We can now state the main theorem of this chapter.

Theorem 4.4 (Main Theorem: Large Components)**.** *Consider the MRCM with connection function ψ having bounded support and subcritical intensity $\lambda \in (0, \lambda_c)$. Then*

$$\frac{|L_1(\xi_\lambda \cap \Lambda_s)|}{\log 2s} \to d \cdot \zeta(\lambda, \psi)^{-1} \text{ in probability},$$

as $s \to \infty$, where L_1 is the largest component.

We can interpret the statement of Theorem 4.4 as saying that the largest component in a box of volume V has on the order of $\log V$ points, where the right factor is exactly the correlation length ζ^{-1}.

To warm up we start with an auxiliary result.

4.2 Auxiliary Result

While the following result is not required to prove the main theorem, it might still be of interest to the reader.

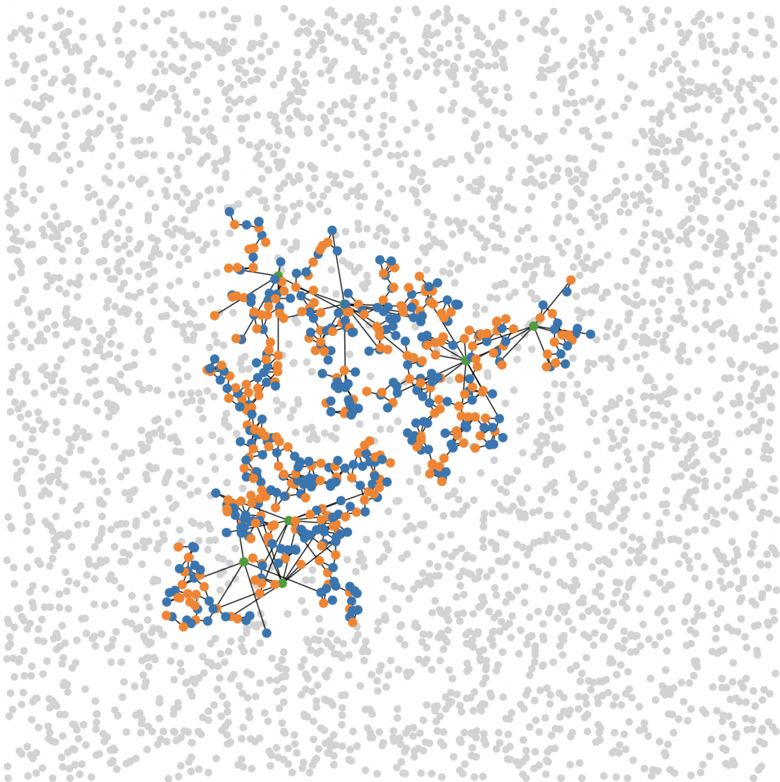

Figure 4-1: The MRCM model in the subcritical regime. Same model as in Figure 1-1, with a smaller intensity λ and a larger observation window. The largest component is highlighted.

Lemma 4.5. *The percolation probability $\theta^m(\lambda)$ is right-continuous in λ for every $m \in \mathbb{M}$.*

It follows immediately from the above lemma that $\theta(\lambda)$ is also right-continuous.

Proof. Let $\varepsilon > 0$. We have the following estimate for all $N \in \mathbb{N}$:

$$1 - \theta^m(\lambda) = \sum_{i\geq 1} \mathbb{P}[|\mathcal{C}_{o_m}| = i \text{ in } \xi_\lambda^{o_m}] \geq \sum_{i=1}^{N} \mathbb{P}[|\mathcal{C}_{o_m}| = i \text{ in } \xi_\lambda^{o_m}].$$

We choose N_m such that $\mathbb{P}[|\mathcal{C}_{o_m}| \leq N_m] > 1 - \theta^m(\lambda) - \varepsilon/2$. Every term in the above sum is differentiable (and therefore continuous) in λ (see [CL24], or [Pen91] for the RCM). Hence, we may choose some $\delta > 0$ so that for every $\lambda' \in (\lambda, \lambda + \delta)$ we find

$$\sum_{i=1}^{N_m} \mathbb{P}[|\mathcal{C}_{o_m}| = i \text{ in } \xi_{\lambda'}^{o_m}] \geq \sum_{i=1}^{N_m} \mathbb{P}[|\mathcal{C}_{o_m}| = i \text{ in } \xi_\lambda^{o_m}] - \varepsilon/2 \geq 1 - \theta^m(\lambda) - \varepsilon.$$

By our definition of N_m,

$$1 - \theta^m(\lambda') = \mathbb{P}[|\mathcal{C}_{o_m}| < \infty \text{ in } \xi_{\lambda'}^{o_m}] \geq \sum_{i=1}^{N_m} \mathbb{P}[|\mathcal{C}_{o_m}| = i \text{ in } \xi_{\lambda'}^{o_m}] > 1 - \theta^m(\lambda) - \varepsilon.$$

It follows that $\theta^m(\lambda) \leq \theta^m(\lambda') < \theta^m(\lambda)+\varepsilon$. Thus, $\theta^m(\lambda)$ is right-continuous everywhere in λ. ■

For $\lambda < \lambda_c$ proving left-continuity is immediate as $\theta^m(\lambda) = 0$. Proving left-continuity at the critical point λ_c is an open question in general. It is equivalent to the absence of the infinite cluster at criticality, which has been solved for certain models in dimensions 2 and dimensions 11 and above. For the MRCM this result has been shown to hold in sufficiently high dimensions [DH22].

Lemma 4.6. *Let $m \in \mathbb{M}$ and $\lambda_0 > \lambda_c$. Then θ^m is left-continuous at λ_0.*

The following proof is adopted from [Dum18].

Proof. We know by [CL24] that the infinite component for the MRCM is unique when it exists. Let $\lambda_0 > \lambda_c$. We want to show that

$$\lim_{\lambda \nearrow \lambda_0} \theta^m(\lambda) = \theta^m(\lambda_0).$$

Now we may create a coupling in λ by sampling our Poisson point process in the space $\mathbb{X} \times \mathbb{R}_{\geq 0}$, with intensity measure $\mathrm{Leb}^{\mathbb{R}^d} \times \rho \times \mathrm{Leb}^{\mathbb{R}_{\geq 0}}$. Then by first restricting to $[0, \lambda]$ and then projecting out the final dimension we recover a MRCM with intensity λ. Now for $\lambda > \lambda_c$ I will write $\mathcal{C}_\lambda^\infty$ to refer to the unique infinite component.

Now assume that $o_m \sim \mathcal{C}_{\lambda_0}^\infty$, but $o_m \nsim \mathcal{C}_\lambda^\infty$ for all $\lambda \in (\lambda_c, \lambda_0)$. For this to be true there would need to exist a Poisson point with value second mark value exactly λ_0, which has probability 0. Hence,

$$\theta^m(\lambda_0) - \lim_{\lambda \nearrow \lambda_0} \theta^m(\lambda) \leq \mathbb{P}[\exists z \in \eta : \pi^{\mathbb{R}_{\geq 0}}(z) = \lambda_0] = 0.$$

Since θ is increasing we find $\theta^m(\lambda_0) = \lim_{\lambda \nearrow \lambda_0} \theta^m(\lambda)$. ■

4.3 Sharpness

Our approach to proving sharpness relies on the method developed in [DT16] for lattice percolation. Similarly to that paper, we define a functional φ_λ that takes a thinning function $f : \mathbb{X} \to [0,1]$ and marks $a, b \in \mathbb{M}$ as an input and returns a real number:

$$\varphi_\lambda(f; a, b) := \lambda \int_{\mathbb{R}^d} (1 - f(x_b))\mathbb{P}[o_a \leftrightarrow x_b \text{ in } \xi[f_*\eta \cup \{o_a, x_b\}]]\mathrm{d}x. \tag{4.1}$$

We can be interpret $\int_{\mathbb{M}} \varphi_\lambda(f; a, b)\rho(\mathrm{d}b)$ as the expected number of points in $\eta \setminus f_*\eta$ that can be reached from the origin with mark a only using points in $f_*\eta$. Indeed, by the Mecke equation, and any measurable subset $M \subset \mathbb{M}$:

$$\int_M \varphi_\lambda(f; a, b)\rho(\mathrm{d}b) = \mathbb{E}\left[\sum_{\substack{x \in \eta \setminus f_*\eta \\ \pi^{\mathbb{M}}(x) \in M}} \mathbf{1}\{o_a \leftrightarrow x \text{ in } \xi[f_*\eta \cup \{o_a\}]\}\right].$$

Let $\mathcal{T}$ be the set of thinning functions with compact support. We can now define a new critical parameter

$$\tilde{\lambda}_c := \sup\left\{\lambda \geq 0 \mid \operatorname*{ess\,sup}_{a \in \mathbb{M}} \inf_{f \in \mathcal{T}} \int_{\mathbb{M}} \varphi_\lambda(f; a, b)\rho(\mathrm{d}b) < 1\right\}.$$

Note that for the RCM it suffices to assume that there exists an $f \in \mathcal{T}$ such that the above quantity is strictly less than 1.

The choice $\operatorname{ess\,sup}_{a \in \mathbb{M}} \inf_{f \in \mathcal{T}} \int_{\mathbb{M}} \varphi_\lambda(f; a, b)\rho(\mathrm{d}b)$ is not immediate. To make sense of it notice that we may view $\varphi_\lambda(f; a, b)$ as the kernel for an operator. For any measurable function $h : \mathbb{M} \to \mathbb{R}$ write:

$$\int_{\mathbb{M}} \varphi_\lambda(f; a, b)h(b)\rho(\mathrm{d}b)$$

Now $\operatorname{ess\,sup}_{a \in \mathbb{M}} \int_M \varphi_\lambda(f; a, b)\rho(\mathrm{d}b) = \|\Phi_f\|_{1,\infty}$. The $1, \infty$-norm is most convenient for our purposes. For more reading on using operators to deal with marks see [CD24]. We will not (explicitly) require operators for the rest of the thesis.

For any $f \in \mathcal{T}$ define the following quantity

$$\Phi_\lambda^{1,\infty}(f) := \operatorname*{ess\,sup}_{a \in \mathbb{M}} \int_{\mathbb{M}} \varphi_\lambda(f; m, b)\rho(\mathrm{d}a). \tag{4.2}$$

Note that $\lambda < \tilde{\lambda}_c$ is equivalent to saying that there exists some $f \in \mathcal{T}$ such that $\Phi_\lambda^{1,\infty}(f) < 1$. Recall the definition of λ_c (Definition 2.9).

Theorem 4.7 (Sharpness)**.** *For any $d \geq 1$ it holds that $\tilde{\lambda}_c = \lambda_c$ and*

(I) For all $\lambda < \lambda_c$ there exists some $c > 0$ such that for all t

$$\operatorname*{ess\,sup}_{m\in\mathbb{M}} \theta_t^m(\lambda,\psi) \le e^{-ct}.$$

(II) For all $\lambda > \lambda_c$ we have

$$\operatorname*{ess\,sup}_{m\in\mathbb{M}} \theta^m(\lambda,\psi) \ge \frac{\lambda-\lambda_c}{\lambda}.$$

Remark 4.8. In the case of the Random Connection Model (i.e. if $|\mathbb{M}| = 1$) we get the following stronger bound:

$$\theta(\lambda) \ge \frac{\lambda-\lambda_c}{\lambda}.$$

Remark 4.9. We note that this theorem implies the original definition of sharpness, i.e. that $\lambda_c = \lambda_c^E$, where $\lambda_c^E := \sup\{\lambda \mid \mathbb{E}[|\mathcal{C}_o|] < \infty\}$. It is immediate that $\lambda_c \ge \lambda_c^E$, as an infinite path with positive probability implies that the expected value of the size of the component of the origin is infinite. The other direction requires a bit more work. By Lemma 4.19 we know that $\mathbb{P}[|\mathcal{C}_o| \ge n]$ decays exponentially in n. Thus,

$$\mathbb{E}[|\mathcal{C}_o|] = \sum_{n\ge1} \mathbb{P}[|\mathcal{C}_o| \ge n] \le \widetilde{C} \sum_{n\ge1} \exp(-\tilde{c}n) < \infty.$$

Remark 4.10. Recall the definition of Z_ψ^∞ from equation (2.3). By using the definition of φ_λ we can recover a standard bound. By choosing $f \equiv 0$ we find

$$\operatorname*{ess\,sup}_{a\in\mathbb{M}} \int_{\mathbb{M}} \varphi_\lambda(f;a,b)\rho(\mathrm{d}b) = \lambda Z_\psi^\infty \quad \text{ and hence } \quad \lambda_c \ge \frac{1}{Z_\psi^\infty}.$$

This shows that for any ψ with $Z_\psi^\infty \in (0,\infty)$ we have $\lambda_c > 0$. If in addition we have $\lambda_c < \infty$, as is the case for $d \ge 2$ (see [CD24, Lemma 2.2]), then the MRCM has a non-trivial phase transition.

We will need the following proposition from [Pen91] (see also [MR96, Proposition 6.2]) which was only proven for the RCM. It was also shown in the proof of [CD24, Lemma 3.4] for the MRCM.

Recall the definition of ψ^A from equation (2.2).

Proposition 4.11. *Let $m \in \mathbb{M}$ and $n \in \mathbb{Z}_{\ge1}$. Let p_n^m be the probability that $\mathbb{P}[|\mathcal{C}_{o_m}| = n]$. Let $g(z_1,\dots,z_k)$ be the probability that $z_1,\dots,z_k$ are in one connected component* *as defined in equation* (3.1)*. Then,*

$$p_{n+1}^m = \frac{\lambda^n}{n!}\int_{\mathbb{X}^n} g(o_m, x_1, \ldots, x_n)\exp\left(-\lambda\int_{\mathbb{X}}\psi^{o_m,\vec{x}}(y)\mathrm{d}y\right)\mathrm{d}\vec{x}.$$

Proof. Notice that we can write

$$p_{n+1}^m = \mathbb{E}\Bigg[\sum_{\{x_1,\ldots,x_n\}\subset\eta}\mathbf{1}\{(o_m, x_1, \ldots, x_n)\text{ connected}\}$$
$$\mathbf{1}\{\{o_m, x_1, \ldots, x_n\}\nsim \eta\setminus\{x_1,\ldots,x_n\}\}\Bigg],$$

where the sum over η explicitly does not contain o. The above equation holds because in the event $|\mathcal{C}_{o_m}| = n+1$ there is exactly one such set $\{x_1, \ldots, x_n\}$ satisfying the stated events. Otherwise, if $|\mathcal{C}_{o_m}| \neq n+1$, such a set does not exist. This corresponds to $n!$ ordered tuples in the factorial measure. Hence, by the Mecke equation

$$p_{n+1} = \frac{\lambda^n}{n!}\int_{\mathbb{X}^n} g(o_m, x_1, \ldots, x_n)\mathbb{P}[\{o_m, x_1, \ldots, x_n\}\nsim\eta]\mathrm{d}\vec{x},$$

To determine if $\{o_m, x_1, \ldots, x_n\}\nsim\eta$, notice that it is equivalent to asking if $|\psi_*^{o_m,\vec{x}}\eta| = 0$. We know by standard Poisson point process theory that $|\psi_*^{o_m,\vec{x}}\eta|$ has a Poisson distribution with intensity $\lambda\int_{\mathbb{X}}\psi^{o_m,\vec{x}}(y)\mathrm{d}y$. Thus, the proposition holds. ∎

Remark 4.12*.* By symmetry, we can also write

$$p_{n+1}^m = \lambda^n\int_{\mathbb{X}^n}\mathbf{1}\{\overline{x}_1 < \cdots < \overline{x}_n\}g(o_m, x_1, \ldots, x_n)\exp\left(-\lambda\int_{\mathbb{X}}\psi^{o_m,\vec{x}}(y)\mathrm{d}y\right)\mathrm{d}\vec{x},$$

where $<$ refers to the lexicographic ordering (although any strict ordering of $\mathbb{R}^d$ would work).

Before we can state the next lemma we need the following definition.

Definition 4.13. For $z\in\mathbb{R}^d$ let $S_z : \mathbf{N}\left((\mathbb{X}\times\mathbb{M})^{[2]}\times[0,1]\right)\to\mathbf{N}\left((\mathbb{X}\times\mathbb{M})^{[2]}\times[0,1]\right)$ be the shift operator which sends each edge in ξ to the same edge translated by z.

We call a function T of the form $T : \mathbb{X}^2\times\mathbf{N}((\mathbb{X}\times\mathbb{M})^{[2]}\times[0,1])\to[0,\infty)$ a *mass transport map*. We call a mass transport map T *shift invariant* if $\mathbb{E}[T(x, y; \xi^{x,y})] = \mathbb{E}[T(x-z, y-z; S_{-z}\xi^{x,y})]$ holds for all $x, y\in\mathbb{X}$ and $z\in\mathbb{R}^d$.

The following is an adaptation of the mass transport principle.

Lemma 4.14 (Mass Transport)**.** *Let* $T : \mathbb{X}^2 \times \mathbf{N}((\mathbb{X}\times\mathbb{M})^{[2]} \times [0,1]) \to [0,\infty)$ *be a shift invariant integrable mass transport map. Then,*

$$\mathbb{E}[\sum_{x\in\eta^o} T(o,x;\xi^o)] = \mathbb{E}[\sum_{x\in\eta^o} T(x,o;\xi^o)]$$

The intuition for the mass transport principle is to view $T(o,x)$ as a function which 'sends mass' from o to x. Therefore, the principle states that all the mass sent out from o must in expectation be equal to all the mass received by o from other points.

For a subset $M \subset \mathbb{M}$ we use the notation o_M to indicate that we sample the mark of o_M uniformly over M relative to ρ, this is equivalent to $\frac{1}{\rho(M)}\int_M \cdot\rho(\mathrm{d}m)$. We further introduce the notation $\mathbb{X}_M$ to denote $\mathbb{R}^d \times M$ for a subset $M \subseteq \mathbb{M}$.

Proof. We start by using the Mecke equation and translation invariance as follows

$$\begin{aligned}\mathbb{E}\left[\sum_{x\in\eta^o} T(o,x;\xi^o)\right] &= \lambda\int_{\mathbb{X}}\int_{\mathbb{M}} \mathbb{E}[T(o_m,x_l;\xi^{o_m,x_l})]\rho(\mathrm{d}m)\mathrm{d}x_l + \mathbb{E}[T(o,o;\xi^o)]\\ &= \lambda\int_{\mathbb{X}}\int_{\mathbb{M}} \mathbb{E}[T(-x_m,o_l;S_{-x}\xi^{-x_m,o_l})]\rho(\mathrm{d}m)\mathrm{d}x_l + \mathbb{E}[T(o,o;\xi^o)].\end{aligned}$$

Note that we have the extra $\mathbb{E}[T(o,o;\xi^o)]$ term since we are using Mecke on η^o.

By substituting $x_l \mapsto -x_l$, Fubini (for swapping the integral over the marks), and swapping the names of the marks, we find that

$$\begin{aligned}\mathbb{E}\left[\sum_{x\in\eta^o} T(o,x;\xi^o)\right] &= \lambda\int_{\mathbb{X}}\int_{\mathbb{M}} \mathbb{E}[T(x_l,o_m;\xi^{x_l,o_m})]\rho(\mathrm{d}m)\mathrm{d}x_l + \mathbb{E}[T(o,o;\xi^o)]\\ &= \mathbb{E}\left[\sum_{x\in\eta^o} T(x,o;\xi^o)\right],\end{aligned}$$

where in the final line we use the Univariate Mecke in the opposite direction. ■

We can use the Mass Transport principle to 'switch marks' in the following way. Going forward the ξ will be dropped from the T notation for compactness.

Corollary 4.15. *Let* $A, B \subseteq \mathbb{M}$ *be measurable with* $\rho(A) > 0$ *and* $\rho(B) > 0$. *Let*

$k \in \mathbb{Z}_{\geq 1}$. *Then*

$$\mathbb{P}[|\mathcal{C}_{o_A}| = k, \mathcal{C}_{o_A} \cap \mathbb{X}_B \neq \varnothing] = \frac{\rho(B)}{\rho(A)} \frac{\mathbb{E}[|\mathcal{C}_{o_B} \cap \mathbb{X}_A| \mid |\mathcal{C}_{o_B}| = k]}{\mathbb{E}[|\mathcal{C}_{o_A} \cap \mathbb{X}_B| \mid |\mathcal{C}_{o_A}| = k, \mathcal{C}_{o_A} \cap \mathbb{X}_B \neq \varnothing]} \mathbb{P}[|\mathcal{C}_{o_B}| = k].$$

Proof. We let A, B and k be as above. We now drop the independent edge marking ξ from the notation of the transport map T for compactness. We choose

$$T(x_m, y_l) = \mathbf{1}\{m \in A\}\mathbf{1}\{l \in B\}\mathbf{1}\{|\mathcal{C}_{x_m}| = k\}\mathbf{1}\{\mathcal{C}_{x_m} \cap \mathbb{X}_B \neq \varnothing\}\mathbf{1}\{x_m \leftrightarrow y_l\}.$$

Then, by the definition of conditional expectation,

$$\begin{aligned}\mathbb{E}\left[\sum_{x\in\eta^o} T(o,x)\right] &= \mathbb{E}\left[\mathbf{1}\{o \in \mathbb{X}_A\}\mathbf{1}\{|\mathcal{C}_o| = k\}\mathbf{1}\{\mathcal{C}_o \cap \mathbb{X}_B \neq \varnothing\} \sum_{x\in\eta^o\cap\mathbb{X}_B} \mathbf{1}\{o \leftrightarrow x\}\right] \\ &= \rho(A)\ \mathbb{P}[|\mathcal{C}_{o_A}| = k, \mathcal{C}_{o_A} \cap \mathbb{X}_B \neq \varnothing]\ \mathbb{E}[|\mathcal{C}_{o_A} \cap \mathbb{X}_B| \mid |\mathcal{C}_{o_A}| = k, \mathcal{C}_{o_A} \cap \mathbb{X}_B \neq \varnothing].\end{aligned}$$

We now perform a similar calculation for $\mathbb{E}\left[\sum_{x\in\eta^o} T(x,o)\right]$.

$$\begin{aligned}&\mathbb{E}\left[\sum_{x\in\eta^o} T(x,o)\right] \\ &= \mathbb{E}\left[\mathbf{1}\{o \in \mathbb{X}_B\} \sum_{x\in\eta^o} \mathbf{1}\{x \in \mathbb{X}_A\}\mathbf{1}\{|\mathcal{C}_x| = k\}\mathbf{1}\{o \leftrightarrow x\}\mathbf{1}\{\mathcal{C}_x \cap \mathbb{X}_B \neq \varnothing\}\right] \\ &= \rho(B)\ \mathbb{P}[|C_{o_B}| = k]\ \mathbb{E}\left[|\mathcal{C}_{o_B} \cap \mathbb{X}_A| \mid \mathbf{1}\{|\mathcal{C}_{o_B}| = k\}\right],\end{aligned}$$

where we use the fact that if o and x are connected then $\mathcal{C}_o = \mathcal{C}_x$. It follows immediately that $\mathcal{C}_{o_B} \cap \mathbb{X}_B$ can never be empty, since it always contains at least the origin. By rearranging the result holds. ■

Remark 4.16. Other results can also be recovered from the Mass Transport by using different choices of $T(x_m, y_l)$. We again assume $\rho(A) > 0$ and $\rho(B) > 0$. If we choose $T(x_m, y_l) = \mathbf{1}\{m \in A, l \in B, x_m \leftrightarrow y_l\}$ we find that

$$\mathbb{E}[|\mathcal{C}_{o_A} \cap \mathbb{X}_B|] = \frac{\rho(B)}{\rho(A)}\mathbb{E}[|\mathcal{C}_{o_B} \cap \mathbb{X}_A|].$$

We can extend this example through additional conditions: $T(x_m, y_l) = \mathbf{1}\{m \in A, l \in$

$B, x_m \leftrightarrow y_l, |C_{x_m}| = k\}$. We now find that

$$\mathbb{P}[|\mathcal{C}_{o_A}| = k] = \frac{\rho(B)}{\rho(A)} \frac{\mathbb{E}[|\mathcal{C}_{o_B} \cap X_A| \mid |\mathcal{C}_{o_B}| = k]}{\mathbb{E}[|\mathcal{C}_{o_A} \cap X_B| \mid |\mathcal{C}_{o_A}| = k]} \mathbb{P}[|C_{o_B}| = k].$$

As a final example, if we choose $T(x_m, y_l) = \mathbf{1}\{m \in A, l \in B, |\mathcal{C}_{x_m}| = k\} \frac{\mathbf{1}\{x_m \leftrightarrow y_l\}}{|\mathcal{C}_{x_m} \cap \mathbb{X}_B|}$, we find that

$$\mathbb{P}[|\mathcal{C}_{o_A}| = k, \mathcal{C}_{o_A} \cap \mathbb{X}_B \neq \varnothing] = \frac{\rho(B)}{\rho(A)} \mathbb{E}\left[\mathbf{1}\{|\mathcal{C}_{o_B}| = k\} \frac{|\mathcal{C}_{o_B} \cap \mathbb{X}_A|}{|\mathcal{C}_{o_B} \cap \mathbb{X}_B|}\right],$$

where we use the convention that $\frac{0}{0} = 0$.

Remark 4.17. In all the above examples (including Corollary 4.15) one may replace all '$= k$' with '$\geq k$'.

4.4 Proof of Sharpness

As stated we take the idea of using φ from [DT16]. This was first done in the continuum by [Zie16] for a large class of bounded hard models. We extend this to bounded soft models.

We will show items (I) and (II) for $\tilde{\lambda}_c$. Then, the fact that $\tilde{\lambda}_c = \lambda$ follows immediately by Remark 4.9. We will make use of the Stopping Set lemma. Let us write

$$\theta_t^m(\lambda) := \mathbb{P}[o_m \leftrightarrow \Lambda_t^c \text{ in } \xi^{o_m}[\eta^{o_m}]],$$

for the t-percolation probability started from mark $m \in \mathbb{M}$.

4.4.1 Item I

Assume $\lambda < \tilde{\lambda}_c$. Then, by definition, we can and do choose a thinning function f such that $\operatorname{ess\,sup}_{a \in \mathbb{M}} \int_M \varphi_\lambda(f; a, b)\rho(\mathrm{d}b) < 1$. Choose $L > 1$ such that $\operatorname{supp} f \subset \Lambda_{L-1}$, so that all vertices of $f_*\eta$ lie in Λ_{L-1}. Let $m \in \mathbb{M}$. For the rest of this section we will write $\mathcal{C}_m := \mathcal{C}(o_m, \xi[f_*\eta \cup \{o_m\}])$. Let C be a finite subset of $\mathbb{X}$, which is a possible candidate for $\mathcal{C}_m$. Given $x \in \mathbb{X}$, remark that $\{o_m \leftrightarrow x \text{ in } \xi[C \cup \{x\}]\} \cap \{\mathcal{C}_m = C\}$ is exactly the same event as $\{x \sim C\} \cap \{\mathcal{C}_m = C\}$, in particular, we remind the reader that we do not resample the edges when we write $\xi[C]$.

We remind the reader of the that we write $\mathrm{d}x$ to refer to $\mathrm{Leb} \otimes \rho(\mathrm{d}(x, a))$, i.e. integration over $\mathbb{X}$.

Let $k \in \mathbb{N}$. The event $\{o_m \leftrightarrow \Lambda_{kL}^c \text{ in } \xi_\lambda^{o_m}\}$ holds if and only if there exists some $x \in \eta \setminus f_*\eta$ such that $o_m \leftrightarrow x$ in $\xi[f_*\eta \cup \{o_m, x\}]$ and $x \leftrightarrow \Lambda_{kL}^c$ off $\mathcal{C}_m$. Then, by applying the Markov inequality followed by the Mecke equation we find that

$$\begin{aligned}
\theta_{kL}^m(\lambda) &\le \mathbb{E}\left[\sum_{x \in \eta \setminus f_*\eta} \mathbf{1}\{o_m \leftrightarrow x \text{ in } \xi^{o_m,x}[f_*\eta \cup \{o_m, x\}]\}\mathbf{1}\{x \leftrightarrow \Lambda_{kL}^c \text{ off } \mathcal{C}_m\}\right] \\
&= \lambda \int_{\mathbb{X}} \mathbb{P}[o_m \leftrightarrow x \text{ in } \xi^{o_m,x}[f_*\eta \cup \{o_m, x\}], x \leftrightarrow \Lambda_{kL}^c \text{ off } \mathcal{C}_m](1 - f(x))\mathrm{d}x \\
&= \lambda \int_{\mathbb{X}} \int \mathbb{P}[o_m \leftrightarrow x \text{ in } \xi^{o_m,x}[C \cup \{x\}], x \leftrightarrow \Lambda_{kL}^c \text{ in } \xi[(\eta \setminus C) \cup \{x\}] \mid \mathcal{C}_m = C] \\
&\qquad \mathbb{P}[\mathcal{C}_m \in \mathrm{d}C](1 - f(x))\mathrm{d}x.
\end{aligned}$$

Notice that when conditioning on $\mathcal{C}_m = C$ the two events become independent. The former depends only on the edges between C and x, while the latter only depends on $\eta \setminus C$ and the edges between itself and x. If $x \notin \Lambda_L$ we have $\mathbb{P}[x \sim C] = 0$. Therefore, for $x \in \Lambda_L$, using the Stopping Set lemma, translation invariance and observing that for a path starting at x to reach Λ_{kL}^c it must first reach $\Lambda_{(k-1)L}(x)^c$ we bound

$$\begin{aligned}
\mathbb{P}[x_a \leftrightarrow \Lambda_{kL}^c \text{ in } \xi[(\eta \setminus C) \cup \{x_a\}] \mid \mathcal{C}_m = C] &\le \mathbb{P}[o_a \leftrightarrow \Lambda_{(k-1)L} \text{ in } \xi^o] \\
&\le \operatorname*{ess\,sup}_{b \in \mathbb{M}} \theta_{(k-1)L}^b(\lambda),
\end{aligned}$$

where we now no longer have a dependence on the mark a. Applying this to the inequality we find that

$$\begin{aligned}
\theta_{kL}^m(\lambda) &\le \lambda \int_{\mathbb{X}} \int \mathbb{P}[o_m \leftrightarrow x_a \text{ in } \xi[C \cup \{x_a\}] \mid \mathcal{C}_m = C] \\
&\qquad \times \mathbb{P}[x_a \leftrightarrow \Lambda_{kL}^c \text{ in } \xi[(\eta \setminus C) \cup \{x_a\}] \mid \mathcal{C}_m = C]\mathbb{P}[\mathcal{C}_m \in \mathrm{d}C](1 - f(x))\mathrm{d}x_a \\
&\le \lambda \int_{\mathbb{X}} \mathbb{P}[o_m \leftrightarrow x_a \text{ in } \xi[\mathcal{C}_m \cup \{x_a\}]](1 - f(x_a))\mathrm{d}x_a \times \operatorname*{ess\,sup}_{b \in \mathbb{M}} \theta_{(k-1)L}^b(\lambda) \\
&= \int_{\mathbb{M}} \varphi_\lambda(f; m, a)\rho(\mathrm{d}a) \times \operatorname*{ess\,sup}_{b \in \mathbb{M}} \theta_{(k-1)L}^b(\lambda).
\end{aligned}$$

The final line is true simply by definition of $\mathcal{C}_m$ and φ_λ. We now take the $\operatorname{ess\,sup}$ over m in order to allow for iteration. We also rewrite using (4.2).

$$\operatorname*{ess\,sup}_{m \in \mathbb{M}} \theta_{kL}^m(\lambda) \le \Phi_\lambda^{1,\infty}(f) \times \operatorname*{ess\,sup}_{b} \theta_{(k-1)L}^b(\lambda).$$

Now we can iterate to find that

$$\operatorname*{ess\,sup}_{m\in\mathbb{M}} \theta^m_{kL}(\lambda) \leq e^{-ck},$$

where $c = -\log(\Phi^{1,\infty}_\lambda(f))$.

Together with the fact that θ_t is decreasing in t we have that item I holds for all $\lambda < \widetilde{\lambda}_c$.

4.4.2 Item II

We will prove item (II) via a differential inequality, again following [DT16]. Let $\lambda > \lambda_c$. We show for all $m \in \mathbb{M}$ and $t \geq 1$ that,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta^m_t(\lambda) \geq \frac{1}{\lambda}\left(\inf_{f\in\mathcal{T}}\int_{\mathbb{M}}\varphi_\lambda(f;m,a)\rho(\mathrm{d}a)\right)(1-\theta^m_t(\lambda)). \tag{4.3}$$

We shall say $u \leftrightarrow v$ *through* x if *every* possible path from u to v passes through x, for some $u, v, x \in \mathbb{X}$. We apply the Stopping Set lemma to the 'outside component'. Write $\mathcal{D}_m = \mathcal{C}(\Lambda^c_t, \xi^{o_m})$ for the outside component (without x). We will denote possible configurations of $\mathcal{C}(\Lambda^c_t, \xi^{o_m})$ by C.

The first equality in the next display comes from the Margulis-Russo's formula. We then marginalize over possible configurations of $\mathcal{D}_m$. Finally, we see that for $o_m \leftrightarrow \Lambda^c_t$ through x we require every path $o_m \leftrightarrow x$ to avoid $\mathcal{D}_m$, and for x to connect to $\mathcal{D}_m$. In symbols

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta^m_t(\lambda,\psi)\\
&= \int_{\Lambda_{t+1}} \mathbb{P}[o_m \leftrightarrow \Lambda^c_t \text{ through } x \text{ in } \xi^{o_m,x}[\eta\cup\{o_m,x\}]]\mathrm{d}x\\
&= \int_{\Lambda_{t+1}}\int \mathbb{P}[o_m \leftrightarrow \Lambda^c_t \text{ through } x \text{ in } \xi^{o_m,x} \mid \mathcal{D}_m = C]\mathbb{P}[\mathcal{D}_m \in \mathrm{d}C]\mathrm{d}x\\
&= \int_{\Lambda_{t+1}}\int \mathbb{P}[o_m \leftrightarrow x \text{ in } \xi^{o_m,x}[(\eta\setminus C)\cup\{o_m,x\}], x \leftrightarrow \Lambda^c_t \text{ in } \xi^x[\eta^x] \mid \mathcal{D}_m = C]\\
&\qquad\qquad \mathbf{1}\{o_m \notin C\}\mathbb{P}[\mathcal{D}_m \in \mathrm{d}C]\mathrm{d}x.
\end{aligned}$$

Next, we observe that $o_m \leftrightarrow x$ and $x \leftrightarrow \Lambda^c_t$ in $\xi^x[\eta^x]$ are conditionally independent, as they rely on a disjoint set of edges. Then we will be able to apply the Stopping Set lemma[1]. We first define

$$\mathcal{T}_t := \{f : \mathbb{X} \to [0,1] \text{ measurable} \mid \operatorname{supp}(f) \subset \Lambda_t\},$$

over which we will take an infimum. For $x \in \mathbb{X}$ let $f^C(x)$ be the probability that $x \nsim C$ and that $x \notin \Lambda_t^C$ so that $(\eta \setminus C \mid \mathcal{D}_m = C) \sim f_*^C \eta$ by the Stopping Set lemma. It holds that $f^C(x) = (1 - \psi^C(x))\mathbf{1}\{x \in \Lambda_t\}$. Hence, for all $x_a \in \Lambda_t$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}[o_m \leftrightarrow \Lambda_t^c] = \int_{\Lambda_{t+1}} \int \mathbb{P}[o_m \leftrightarrow x_a \text{ in } \xi^{o_m, x_a}[(\eta \setminus C) \cup \{o_m, x_a\}] \mid \mathcal{D}_m = C] \\ \times \mathbb{P}[x_a \leftrightarrow \Lambda_t^c \text{ in } \xi^{x_a}[\eta^{x_a}] \mid \mathcal{D}_m = C]\mathbf{1}\{o_m \notin C\}\mathbb{P}[\mathcal{D}_m \in \mathrm{d}C]\mathrm{d}x_a\end{aligned}$$

We will now be able to apply the Stopping Set lemma. Notice that

$$\mathbb{P}[x \leftrightarrow \Lambda_t^c \text{ in } \xi^x[\eta^x] \mid \mathcal{D}_m = C] = \mathbb{P}[\{x \sim C\} \cup \{x \in \Lambda_t^c\}] = 1 - f^C(x).$$

Combining we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}[o_m \leftrightarrow \Lambda_t^c] \\ &= \int_{\Lambda_{t+1}} \int \mathbb{P}[o_m \leftrightarrow x_a \text{ in } \xi[f_*^C \eta \cup \{o_m, x_a\}]](1 - f^C(x_a))\mathbf{1}\{o_m \notin C\}\mathbb{P}[\mathcal{D}_m \in \mathrm{d}C]\mathrm{d}x_a \\ &\geq \inf_{f \in \mathcal{T}_t} \int_{\Lambda_{t+1}} \int \mathbb{P}[o_m \leftrightarrow x_a \text{ in } \xi[f_* \eta \cup \{o_m, x_a\}]](1 - f(x_a))\mathbf{1}\{o_m \notin C\}\mathbb{P}[\mathcal{D}_m \in \mathrm{d}C]\mathrm{d}x_a.\end{aligned}$$

In the last line we may take the infimum over $\mathcal{T}_t$ since $\operatorname{supp}(f^C) \subset \Lambda_t$ and so $f^C \in \mathcal{T}_t$.

We can expand the domain of integration to all of $\mathbb{R}^d$, since the integrand is equal to zero outside of Λ_{t+1}. Since $\mathcal{T}_t \subset \mathcal{T}$, we can bound the $\inf_{f \in \mathcal{T}_t}$ by $\inf_{f \in \mathcal{T}}$ from below.

[1] The Stopping set lemma still holds as expected, the only required modification is to start 'growing' with the random set $A_0 = \eta \cap \Lambda_t^c \cap \Lambda_{t+1}$. Another way to formalize this is by introducing a virtual 'ghost-vertex' with a modified connection function such that it connects to all points in Λ_t^c.

We explicitly write out the mark of $x = x_a$. We find that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}[o_m \leftrightarrow \Lambda_t^c] &\geq \frac{1}{\lambda}\inf_{f\in\mathcal{T}_t}\lambda\int_{\mathbb{X}}\mathbb{P}[o_m \leftrightarrow x \text{ in } \xi^{o_m,x_a}[f_*\eta\cup\{o_m,x_a\}]](1-f(x_a))\mathrm{d}x_a \\
&\qquad\qquad \times\int \mathbf{1}\{o_m\notin C\}\mathbb{P}[\mathcal{D}_m\in \mathrm{d}C] \\
&\geq \frac{1}{\lambda}\inf_{f\in\mathcal{T}}\int_{\mathbb{M}}\varphi_\lambda(f;m,a)\rho(\mathrm{d}a)\int \mathbf{1}\{o_m\notin C\}\mathbb{P}[\mathcal{D}_m\in\mathrm{d}C] \qquad (4.4)\\
&= \frac{1}{\lambda}\inf_{f\in\mathcal{T}}\int_{\mathbb{M}}\varphi_\lambda(f;m,a)\rho(\mathrm{d}a)\mathbb{P}[o_m \not\leftrightarrow \Lambda_t^c],
\end{aligned}
$$

where to get (4.4) from the previous line we apply the definition of φ_λ (4.1) and take the inf over the larger set $\mathcal{T}$. This gives us (4.3). For the RCM we could now continue to solve the differential inequality. However, $\lambda > \tilde{\lambda}_c$ only guarantees that $\operatorname{ess\,sup}_{m\in\mathbb{M}}\inf_{f\in\mathcal{T}}\int_{\mathbb{M}}\varphi_\lambda(f;m,a)\rho(\mathrm{d}a)\geq 1$ for all f.

We define

$$M_\lambda(t) := \left\{m\in\mathbb{M} \mid \inf_{f\in\mathcal{T}}\int_{\mathbb{M}}\varphi_\lambda(f;m,a)\rho(\mathrm{d}a)\geq t\right\}.$$

We find that $\rho(M_\lambda(1)) > 0$ as follows. First note that by assumption $\lambda/\tilde{\lambda}_c > 1$. By definition of φ_λ (4.1) it holds for all $a, m\in\mathbb{M}$ that $\varphi_\lambda(\cdot\,; m,a)\geq \frac{\lambda}{\tilde{\lambda}_c}\varphi_{\tilde{\lambda}_c}(\cdot\,;m,a)$.

Hence,

$$\operatorname*{ess\,sup}_{m\in\mathbb{M}}\inf_{f\in\mathcal{T}}\int_{\mathbb{M}}\varphi_\lambda(f;m,a)\rho(\mathrm{d}a)\geq\frac{\lambda}{\tilde{\lambda}_c}\operatorname*{ess\,sup}_{m}\inf_{f}\int_{\mathbb{M}}\varphi_{\tilde{\lambda}_c}(f;m,a)\rho(\mathrm{d}a)\geq\frac{\lambda}{\tilde{\lambda}_c}>1.$$

And so the claim that $\rho(M_\lambda(1)) > 0$ holds.

We now derive item (II) from (4.3). Let $m\in M_\lambda(1)$. Thus, for all $f\in\mathcal{T}$ we have $\varphi_\lambda(f)\geq 1$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_t^m(\lambda)\geq\frac{1}{\lambda}(1-\theta_t^m(\lambda)).$$

We divide both sides by $1-\theta_t^m$ and integrate from $\tilde{\lambda}_c$ to λ to obtain

$$\int_{\tilde{\lambda}_c}^{\lambda}\frac{\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_t^m(\lambda')}{1-\theta_t^m(\lambda')}\mathrm{d}\lambda'\geq\int_{\tilde{\lambda}_c}^{\lambda}\frac{1}{\lambda'}\mathrm{d}\lambda'.$$

By u-substitution with $u = 1-\theta_t^m(\lambda')$ we recover

$$-\log(1-\theta_t^m(\lambda))+\log(1-\theta_t^m(\tilde{\lambda}_c))\geq\log(\frac{\lambda}{\tilde{\lambda}_c}).$$

By rearranging we get $\tilde{\lambda}_c(1-\theta_t^m(\tilde{\lambda}_c)) \geq \lambda(1-\theta_t^m(\lambda))$, and hence

$$\theta_t^m(\lambda) \geq \frac{\lambda - \tilde{\lambda}_c(1-\theta_t^m(\tilde{\lambda}_c))}{\lambda}.$$

Now we can let $t \to \infty$ and using the fact that $(1-\theta^m(\tilde{\lambda}_c)) \leq 1$ we recover

$$\theta^m(\lambda) \geq \frac{\lambda - \tilde{\lambda}_c}{\lambda}.$$

It holds that $\operatorname{ess\,sup}_{m\in M_\lambda(1)} \theta^m(\lambda) \leq \operatorname{ess\,sup}_{m\in\mathbb{M}} \theta^m(\lambda)$, and so

$$\operatorname*{ess\,sup}_{m\in\mathbb{M}} \theta^m(\lambda) \geq \frac{\lambda - \tilde{\lambda}_c}{\lambda}.$$

Additionally, since $\rho(M_\lambda(1)) > 0$ we get $\theta(\lambda) > 0$. ■

4.5 Proof of Theorem 4.4

We now have the tools to show Theorem 4.4. Remember we are trying to show for $\lambda \in (0, \lambda_c)$ that

$$\frac{|L_1(\xi_\lambda \cap \Lambda_t)|}{\log t} \longrightarrow \frac{d}{\zeta(\lambda)} \quad \text{in probability.}$$

We will first use exponential decay of the t-percolation probability to prove exponential decay of the number of vertices in the component containing the origin. We then use this fact to show $\zeta(\lambda)$ is well defined, continuous and decreasing.

4.5.1 Exponential decay in volume

To get a hold of the relevant facts about the inverse correlation length, we first need an exponential bound on the number of vertices in the component containing the origin. We adapt the following arguments from [Dum18] to the MRCM. We start with a simple lemma.

Lemma 4.18. *Suppose $\lambda < \lambda_c$. Let $b \geq a \geq 2$. Then*

$$\mathbb{P}_\lambda[\Lambda_a \leftrightarrow \Lambda_b^c] \leq \lambda(2a)^d \exp(-c(b-a))$$

where $c = c(\lambda)$ is the same constant as in Theorem 4.7.

Proof. We apply the Markov inequality followed by the Mecke equation. We find that

$$\begin{aligned}
\mathbb{P}_\lambda[\Lambda_a \leftrightarrow \Lambda_b^c] &\le \mathbb{E}_\lambda\left[\sum_{x\in\mathcal{P}_\lambda(\Lambda_a)} \mathbf{1}\{x \leftrightarrow \Lambda_b^c\}\right] \\
&= \lambda \int_{\Lambda_a} \mathbb{P}[x \leftrightarrow \Lambda_b^c]\mathrm{d}x \\
&\le \lambda \mathrm{Leb}(\Lambda_a)\mathbb{P}_\lambda[o \leftrightarrow \Lambda_{b-a}^c] \\
&\le \lambda(2a)^d \exp(-c(b-a)),
\end{aligned}$$

where the exponential bound comes from Theorem 4.7. ■

Lemma 4.19 (Exponential decay of $|\mathcal{C}_o|$)**.** *For all $\lambda \in (0, \lambda_c)$ there exist constants $\tilde{C}, \tilde{c} > 0$ such that for all $m \in \mathbb{M}$ and all n sufficiently large we find*

$$\mathbb{P}[|\mathcal{C}_{o_m}| \ge n] \le \tilde{C}\exp(-\tilde{c}n), \tag{4.5}$$

uniformly over m.

Proof. We follow the proof given in [Dum18, Theorem 3.7]. We need to make some changes to accommodate the Poisson point process. In particular, we upper bound the number of Poisson points in a random subset of $\mathbb{R}^d$. Recall the notation $\Lambda_\kappa(x) := x + [-\kappa, \kappa]^d$. We will not explicitly write out the mark for the origin o_m.

Let $\kappa > 4$, to be chosen later. We define a new graph which we will call $\mathcal{H}_\kappa$ with vertex set $2\kappa\mathbb{Z}^d$ and edges between vertices $x, y \in \mathcal{H}_\kappa$ if and only if $\|x-y\|_\infty \le 2\kappa$; this graph has degree $D = 3^d - 1$, where in particular D is independent of κ.

We call a finite connected set of vertices of $\mathcal{H}_\kappa$ an *animal.* We denote by $\mathcal{A}(k)$ the set of animals that contain the origin and are of cardinality k. For each animal $A \in \mathcal{A}(k)$ let $T(A)$ be a maximal stable set of sites in A. That is the largest collection of $x \in A$ such that no two $x, y \in T(A)$ share an edge. In the case of multiple possible such sets any tiebreaker will work.

We shall say that a vertex x in $\mathcal{H}_\kappa$ is *good* if the event $\{\Lambda_\kappa(x) \leftrightarrow \Lambda_{3\kappa/2}(x)^c \text{ in } \xi_\lambda^o\}$ occurs. We shall say an animal A is good if every $x \in A$ is good.

Let $c' \in (0,1)$, to be chosen later. We set $k := k(n) := \lfloor \frac{c'n}{\lambda(2\kappa)^d} \rfloor$. Let F_k be the event that there exists some $A \in \mathcal{A}(k)$ such that A is good. Then

$$\mathbb{P}[|\mathcal{C}_o| \geq n] \leq \mathbb{P}[F_k] + \mathbb{P}[|\mathcal{C}_o| \geq n, F_k^c]. \tag{4.6}$$

First we bound $\mathbb{P}[|C_o| \geq n, F_k^c]$. We define $C_o^* := \{x \in \kappa\mathbb{Z}^d \mid \Lambda_\kappa(x) \cap \mathcal{C}_o \neq \varnothing\}$, to be the minimal animal that contains $\mathcal{C}_o$.

Then C_o^* is connected, $o \in C_o^*$ and if $|C_o^*| > D+1$, then all sites in C_o^* are good. It follows that if $k \geq D+1$ and $|C_o^*| \geq k$ then F_k occurs. Also, if $|C_o^*| < k$ then there exists at least one animal $A \in \mathcal{A}(k)$ with $C_o^* \subset A$. Hence, if n is large enough so that $k \geq D+1$, then

$$\begin{aligned}
\mathbb{P}[|\mathcal{C}_o| \geq n, F_k^c] &\leq \mathbb{P}[|\mathcal{C}_o| \geq n, |C_o^*| < k] \\
&\leq \mathbb{P}\left[\bigcup_{A \in \mathcal{A}(k)} \left\{\eta\big(\cup_{x \in A} \Lambda_\kappa(x)\big) \geq n\right\}\right] \\
&\leq \sum_{A \in \mathcal{A}(k)} \mathbb{P}[\mathrm{Pois}(\lambda\kappa^d|A|) \geq n].
\end{aligned}$$

By [Pen03, Lemma 9.3] we know that $|\mathcal{A}(k)| \leq 2^{kD}$.

Hence,

$$\mathbb{P}[|\mathcal{C}_o| \geq n, F_{k(n)}^c] \leq 2^{kD}\mathbb{P}[\mathrm{Pois}(\lambda k\kappa^d) \geq n] \leq 2^{kD}\mathbb{P}[\mathrm{Pois}(c'n) \geq n].$$

Assume $c' \leq e^{-4}$. Then by [Pen03, Lemma 1.2, eq. (1.12)] (see also Section 4.6) we have that $\mathbb{P}[\mathrm{Pois}(c'n) \geq n] \leq e^{-2n}$.

Assume also that $c' \leq \lambda\kappa^d/(D\log 2)$, then

$$2^{kD} \leq \exp((D\log 2)c'n/(\lambda\kappa^d)) \leq e^n.$$

Combining these estimates yields

$$\mathbb{P}[|\mathcal{C}_o| \geq n, F_k^c] \leq \exp(-n). \tag{4.7}$$

Next we bound $\mathbb{P}[F_k]$. It is always possible to find a set $T(A)$ of cardinality at least $(k-1)/D$. Such a set can be constructed iteratively by using a 'greedy' approach; subsequently adding vertices adjacent to neighbors of already added vertices in such a way there they are not a direct neighbor of any already chosen vertex.

Then by the union bound,

$$\begin{aligned}\mathbb{P}[F_k] &\leq \sum_{A\in\mathcal{A}(k)} \mathbb{P}[A \text{ is good}] \\ &\leq \sum_{A\in\mathcal{A}(k)} \mathbb{P}[\forall x \in T(A) : x \text{ is good}]\end{aligned}$$

We use the fact that the goodness of sites in $T(A)$ are independent, since the boxes that define them do not intersect.

$$\begin{aligned}\mathbb{P}[F_k] &\leq \sum_{A\in\mathcal{A}(k)} \mathbb{P}[\Lambda_\kappa \leftrightarrow \Lambda^c_{3\kappa/2}]^{|T(A)|} \\ &\leq |\mathcal{A}(k)|\mathbb{P}[\Lambda_\kappa \leftrightarrow \Lambda^c_{3\kappa/2}]^{k/D}.\end{aligned}$$

We can now apply Lemma 4.18 and pick $\kappa > 2$ sufficiently large such that

$$\mathbb{P}[\Lambda_\kappa \leftrightarrow \Lambda^c_{3\kappa/2}] \leq \lambda\kappa^d \exp(-c\kappa/2) \leq \frac{1}{e} \cdot 2^{-D^2}.$$

Using $|A(k)| \leq 2^{Dk}$ again, we thus find that

$$\mathbb{P}[F_k] \leq 2^{Dk}(e^{-k/D}2^{-kD}) \leq \exp(-c'n/(2\lambda\kappa^d D)). \tag{4.8}$$

Putting (4.6), (4.7) and (4.8) together gives us (4.5). ∎

4.5.2 Properties of the inverse correlation length

To prove Theorem 4.4, the only missing ingredient is a more detailed understanding of the inverse correlation length ζ. We recall the Definition 4.2.

$$\begin{aligned}\zeta^m(\lambda) &:= \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}[|\mathcal{C}_{o_m}| = n \text{ in } \xi^{o_m}_\lambda] && (\mathbf{C1}) \\ &= \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}[n \leq |\mathcal{C}_{o_m}| < \infty \text{ in } \xi^{o_m}_\lambda]. && (\mathbf{C2})\end{aligned}$$

We start with showing that the limit exists using the following lemma, a proof of which can be found in [Fek23].

Lemma 4.20 (Fekete's Subadditivity Lemma). *Let $(u_n)_n \subset \mathbb{R}$ be a sequence of num-*

bers. If for all n and m we have $u_{n+m} \leq u_n + u_m$, then

$$\lim_{n\to\infty} \frac{u_n}{n} = \inf_n \frac{u_n}{n} \in [-\infty, \infty).$$

The following proof is based on [Pen03], where the same fact is shown for the case where $\psi(x) = \mathbf{1}\{\|x\| \leq 1\}$.

Proof of Lemma 4.3. First we show convergence of (**C1**). We then show equality of (**C1**) and (**C2**). Then we show continuity and the limiting behavior as λ goes to zero.

We show each of these properties point-wise in m.

Convergence To prove convergence we use Fekete's Subadditivity Lemma. Showing subadditivity for $-\log p_n^m$ is equivalent to showing supermultiplicativity for p_n. It is easier to show supermultiplicativity for the following modified quantity

$$\tilde{p}_{n+1}^m = \lambda^n \int_{\mathbb{X}^n} \mathbf{1}\{o_m < \overline{x}_1 < \cdots < \overline{x}_n\} g(o_m, x_1, \ldots, x_n) \exp\left(-\lambda \int_{\mathbb{X}} \psi^{o_m, \vec{x}}(z) \mathrm{d}z\right) \mathrm{d}\vec{x},$$

where the difference to p_{n+1} is that the indicator includes the origin[2]. Note that g is defined as in Proposition 4.11. It can be interpreted as the probability that $|C_{o_m}| = n+1$ *and* that the origin is the left-most point. It holds that $\tilde{p}_{n+1} = p_{n+1}/(n+1)$. This is true since we have a uniform (i.e. $\frac{1}{n+1}$) chance that the origin is the left-most vertex. Notice that by symmetry we also have:

$$\begin{aligned}\tilde{p}_{n+1}^m = \lambda^n \int_{\mathbb{X}^n} &\mathbf{1}\{\overline{x}_1 < \cdots < \overline{x}_i < o_m < \overline{x}_{i+1} < \cdots < \overline{x}_n\} \\ &\times g(o_m, x_1, \ldots, x_n) \exp\left(-\lambda \int_{\mathbb{X}} \psi^{o_m, \vec{x}}(z) \mathrm{d}z\right) \mathrm{d}\vec{x},\end{aligned}$$

for every $i \in [\![1, n]\!]$. Combining we find

$$\begin{aligned}&\tilde{p}_{n+1}^m \tilde{p}_{k+1}^m \\ &= \lambda^{n+k} \int_{\mathbb{X}^n} \int_{\mathbb{X}^k} g(o_m, x_1, \ldots, x_n) g(o_m, y_1, \ldots, y_m) \mathbf{1}\{o_m < \overline{x}_1 < \cdots < \overline{x}_n\} \\ &\quad \times \mathbf{1}\{\overline{y}_1 < \cdots < \overline{y}_k < o_m\} \exp\left(-\lambda \int_{\mathbb{X}} \psi^{o_m, \vec{x}}(z) \mathrm{d}z + \int_{\mathbb{X}} \psi^{o_m, \vec{y}}(z) \mathrm{d}z\right) \mathrm{d}\vec{y} \mathrm{d}\vec{x}\end{aligned}$$

[2] When written as in Remark 4.12

It is immediately clear that

$$\mathbf{1}\{\overline{y}_1 < \cdots < \overline{y}_k < o_m\}\mathbf{1}\{o_m < \overline{x}_1 < \cdots < \overline{x}_n\} = \mathbf{1}\{\overline{y}_1 < \cdots < \overline{y}_k < o_m < \overline{x}_1 < \cdots < \overline{x}_n\}.$$

We rename $y_1, \ldots, y_m$ to $x_{n+1}, \ldots, x_{n+m}$. We now reduced the problem to showing

$$g(o_m, x_1, \ldots, x_n)g(o_m, x_{n+1}, \ldots, x_{n+m}) \leq g(o_m, x_1, \ldots, x_{n+m})$$

and

$$\psi^{o_m,x_1,\ldots,x_n}(z) + \psi^{o_m,x_{n+1},\ldots,x_{n+m}}(z) \geq \psi^{o,\vec{x}}(z).$$

The first equation follows, since the union of two connected graphs with a vertex in common is again a connected graph. The second equation holds by the union bound. Hence, we get

$$\tilde{p}^m_{n+1}\tilde{p}^m_{k+1} \leq \tilde{p}^m_{n+k+1}.$$

By choosing $\tilde{u}^m_n := -\log \tilde{p}_{n+1}$ and applying Fekete's Subadditivity Lemma we know that $\lim_{n\to\infty} \frac{u^m_n}{n}$ exists. By rearranging and using the fact that $\tilde{p}^m_n = \frac{p^m_n}{n}$ we find that

$$\lim_{n\to\infty} -\frac{\log \tilde{p}^m_n}{n} = \lim_{n\to\infty} -\frac{\log p^m_n - \log n}{n} = \lim_{n\to\infty} -\frac{\log p^m_n}{n}.$$

Furthermore, since the u^m_n's are lower bounded by 0 we get the stronger bound that $\zeta^m(\lambda) \in [0, \infty)$. We also find immediately by Lemma 4.19 that if $\lambda < \lambda_c$ we have $\zeta^m(\lambda) > 0$:

$$-\frac{1}{n}\log(\tilde{C}\exp(-\tilde{c}n)) \geq \tilde{c} - \frac{\log \tilde{C}}{n} \xrightarrow{n\to\infty} \tilde{c} > 0.$$

Importantly, the above bound holds uniformly for every m, so we find that

$$\zeta(\lambda) \geq \zeta^{\min}(\lambda) > 0. \tag{4.9}$$

Equivalence of definitions (C1) and (C2) We show the two definitions of the m-inverse correlation length are indeed equivalent. It suffices to show that

$$q^m_n := q^m_n(\lambda) := \left(\frac{\mathbb{P}[|\mathcal{C}_{o_m}| = n]}{\mathbb{P}[n \leq |\mathcal{C}_{o_m}| < \infty]}\right)^{\frac{1}{n}} \xrightarrow{n\to\infty} 1$$

We immediately know that $q^m_n \leq 1$, so we only need to bound it from below. Let $0 < \zeta^- < \zeta^m(\lambda) < \zeta^+$, to be chosen later (which we can do by (4.9)). For now $\zeta^m(\lambda)$ will refer to (**C1**), i.e. the definition using $\mathbb{P}[|\mathcal{C}_{o_m}| = n]$. Then, by existence of the limit, we know that for n sufficiently large it holds that

$$e^{-n\zeta^+} < \mathbb{P}[|\mathcal{C}_{o_m}| = n] < e^{-n\zeta^-}. \tag{4.10}$$

This gives us the following bounds on $\mathbb{P}[n \le |\mathcal{C}_{o_m}| < \infty] = \sum_{k\ge n} \mathbb{P}[|\mathcal{C}_{o_m}| = n]$,

$$\frac{e^{-\zeta^+ n}}{1-e^{-\zeta^+}} < \mathbb{P}[n \le |\mathcal{C}_{o_m}| < \infty] < \frac{e^{-\zeta^- n}}{1-e^{-\zeta^-}}. \tag{4.11}$$

Now let $\varepsilon > 0$. We want to show the existence of an n_0 such that for all $n \ge n_0$ we have $q_n^m \in (1-\varepsilon, 1]$. We now choose $\zeta^\pm = \zeta(\lambda) \pm \varepsilon/4$. Let n be sufficiently large, such that (4.10) holds and

$$e^{-\varepsilon/2}(1-e^{-\zeta^-})^{\frac{1}{n}} > (1-\varepsilon). \tag{4.12}$$

Now we bound q_n^m using (4.10) and (4.11):

$$q_n^m > \frac{e^{-\zeta^+}(1-e^{-\zeta^-})^{\frac{1}{n}}}{e^{-\zeta^-}}.$$

Substituting for the defintion of $\zeta^\pm$ together with (4.12) we find that

$$q_n^m > 1-\varepsilon.$$

Thus, the two definitions of ζ^m are equivalent.

Continuity and monotonicity For continuity and monotonicity of ζ^m we follow [Pen03, Theorem 10.1]. First we show that ζ^m is non-increasing and continuous. Consider the quantities $q_n(\lambda) = \mathbb{P}[|\mathcal{C}_{o_m}| = n]$ and $q_n^+ := \mathbb{P}[n \le |\mathcal{C}_{o_m}| < \infty]$. It is easy to see that q_n^+ is increasing in λ for every n in the subcritical regime. We now define

$$u^m(\lambda) := \lim_{n\to\infty} q_n^+(\lambda)^{1/n} = e^{-\zeta^m(\lambda)}.$$

We see that u^m is non-decreasing, which in turn shows that ζ^m is non-increasing in λ.

We move on to continuity in λ. We couple the MRCM at different intensities λ. Consider $0 < \lambda < \mu < \lambda_c$. We mark every point X_i with an additional mark $\lambda_i \sim \mathrm{Unif}([0,\mu])$. For a given intensity λ we may retain all points where $\lambda_i \le \lambda$ to recover ξ_λ from ξ_μ.

Now, one way for the event $\{|\mathcal{C}_{o_m}| = n \text{ in } \xi_\lambda^{o_m}\}$ to hold is to require $\{|\mathcal{C}_{o_m}| = n \text{ in } \xi_\mu^{o_m}\}$ and all $(X_i, \lambda_i) \in \mathcal{C}_{o_m}$ have the property that $\lambda_i < \lambda$, which has probability λ/μ, per vertex. This gives

$$\mathbb{P}[|\mathcal{C}_{o_m}| = n \text{ in } \xi_\lambda^{o_m}] \geq \left(\frac{\lambda}{\mu}\right)^{n-1} \mathbb{P}[|\mathcal{C}_{o_m}| = n \text{ in } \xi_\mu^{o_m}].$$

Then

$$u^m(\lambda) = \lim_{n\to\infty} q_n(\lambda)^{1/n} \geq \lim_{n\to\infty} \left(\left(\frac{\lambda}{\mu}\right)^{n-1} q_n^m(\mu)\right)^{1/n} = \frac{\lambda}{\mu} u^m(\mu).$$

This together with the fact that u^m is non-decreasing gives the continuity for u^m as follows. Fix some $\lambda \in (0, \lambda_c)$. Let $\varepsilon > 0$, $\delta = \lambda\varepsilon/u^m(\lambda)$ and $\lambda' \in (\lambda - \delta, \lambda + \delta)$. Assume $\lambda' > \lambda$. Then

$$u^m(\lambda') - u^m(\lambda) \leq \frac{\lambda'}{\lambda} u^m(\lambda) - u^m(\lambda) \leq u^m(\lambda)\frac{\delta}{\lambda} \leq \varepsilon.$$

The case where $\lambda' < \lambda$ is analogous.

Continuity of u^m implies that ζ^m is also continuous in λ in the range $(0, \lambda_c)$. Notice further that our choice of ε and δ did not depend on m. Thus, the family $(\zeta^m)_{m\in\mathbb{M}}$ is equicontinuous implying that ζ is also continuous.

Near zero behavior We show as $\lambda \to 0$ that $\zeta^m(\lambda, \psi) \to \infty$. We show this by bounding the size of the component of the origin by the total progeny (total number of vertices) of a Galton-Watson tree. For more details see [Pen93] where a similar strategy was used. We use the following theorem by [Dwa69].

Lemma 4.21 (Dwass' formula)**.** *Let τ be a Galton-Watson tree with offspring distribution ν. Let $|\tau|$ be its total progeny. We let $N_k := X_1 + \cdots + X_k$ where $X_j \sim \nu$ iid. Then*

$$\mathbb{P}[|\tau| = k] = \frac{1}{k}\mathbb{P}[N_k = k - 1].$$

Now we can dominate $|\mathcal{C}_{o_m}|$ by a Galton-Watson tree τ with offspring distribution $\mathrm{Pois}(\lambda Z_\psi^\infty)$, where $Z_\psi^\infty = \operatorname{ess\,sup}_{a\in\mathbb{M}} \int_{\mathbb{R}^d}\int_{\mathbb{M}} \psi(x; a, b)\rho(\mathrm{d}b)$. It dominates $|\mathcal{C}_{o_m}|$ for all m in the sense that

$$\mathbb{P}[|\tau| \geq k] \geq \mathbb{P}[|\mathcal{C}_{o_m}| \geq k].$$

The use of the essential supremum will help us avoid having to track the marks of all vertices in $\mathcal{C}_{o_m}$. For simplicity, we write $\alpha = \lambda Z_\psi^\infty$.

A proof of this for the RCM was given in [Pen93]. We know that for $\alpha < 1$ the Galton-Watson tree is subcritical and so the total progeny $|\tau|$ of the tree is almost-surely finite.

By using Dwass' formula we find that

$$
\begin{aligned}
\mathbb{P}[|\tau| \geq k] &= \sum_{j=k}^{\infty} \frac{1}{j} \mathbb{P}[\mathrm{Pois}(j\alpha) = j-1] \\
&= \sum_{j=k}^{\infty} \frac{e^{-j\alpha}(j\alpha)^{j-1}}{j!}.
\end{aligned}
$$

By the fact that for all $j \in \mathbb{N}$ we have $\frac{j^j}{j!} \leq e^j$ we find that

$$
\begin{aligned}
\mathbb{P}[|\tau| \geq k] &\leq \frac{1}{\alpha} \sum_{j=k}^{\infty} (e^{(1-\alpha)}\alpha)^j \\
&= \frac{(e^{1-\alpha}\alpha)^k}{\alpha(1-e^{1-\alpha}\alpha)}.
\end{aligned}
$$

By substituting λZ_ψ^∞ back in for α we find that

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}[|C_o| \geq n] &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}[|\tau| \geq n] \\
&\leq \limsup_{n\to\infty} \Big((1-\alpha) + \log \alpha - \frac{1}{n} \log \alpha - \frac{1}{n} \log(1 - e^{1-\alpha}\alpha) \Big) \\
&\leq 1 - \lambda Z_\psi^\infty + \log \left(\lambda Z_\psi^\infty \right).
\end{aligned}
$$

And so the upper bound tends to $-\infty$ as λ (or Z_ψ^∞) goes to zero. Hence, by applying Definition 4.2, we have that $\zeta^m(\lambda, \psi) \to \infty$.

Equivalence of definitions (C3) and (C4) Let $\zeta^{\min}(\lambda) := \operatorname{ess\,inf}_{m\in\mathbb{M}} \zeta^m$. We know that $\zeta^{\min}(\lambda) > 0$. Let $\varepsilon > 0$ be small. Define $M_\varepsilon := \{m \in \mathbb{M} \mid \zeta^m(\lambda) < \zeta^{\min}(\lambda) + \varepsilon\}$. We know that $\rho(M_\varepsilon) > 0$ by the definition of $\operatorname{ess\,inf}$.

For $m \in \mathbb{M}$, by the definition of ζ^m, and for n sufficiently large it holds that:

$$
e^{-(\zeta^m+\varepsilon)n} \leq \mathbb{P}[|\mathcal{C}_{o_m}| = n] \leq e^{-(\zeta^m-\varepsilon)n}. \tag{4.13}
$$

And so we can upper bound

$$
\int_{\mathbb{M}} \mathbb{P}[|\mathcal{C}_{o_m}| = n] \rho(\mathrm{d}m) \leq \int_{\mathbb{M}} e^{-(\zeta^m-\varepsilon)n} \rho(\mathrm{d}m) \leq e^{-(\zeta^{\min}-\varepsilon)n}.
$$

For the lower bound we know by definition of M_ε and (4.13):

$$\int_{\mathbb{M}} \mathbb{P}[|\mathcal{C}_{o_m}| = n]\rho(\mathrm{d}m) \geq \int_{M_\varepsilon} \mathbb{P}[|\mathcal{C}_{o_m}| = n]\rho(\mathrm{d}m) \geq e^{-(\zeta^{\min}+2\varepsilon)n}\rho(M_\varepsilon).$$

Now, taking the $-\log$ of these expressions and dividing by n we find that:

$$-\frac{1}{n}\log \int_{\mathbb{M}} \mathbb{P}[|\mathcal{C}_{o_m}| = n]\rho(\mathrm{d}m) \leq \zeta^{\min} + 2\varepsilon - \frac{1}{n}\log(\rho(M_\varepsilon)) \xrightarrow{n\to\infty} \zeta^{\min} + 2\varepsilon.$$

Thus, it follows that $\zeta(\lambda) = \zeta^{\min}(\lambda)$.

All marks are equal Now assume that $A, B \subset \mathbb{M}$ measurable with $\rho(A) > 0$ and $\rho(B) > 0$. Assume by contradiction that $\zeta^A > \zeta^B$. We write $\zeta^A - \zeta^B = \delta > 0$. Then, by definition of ζ there must exist some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that

$$-\frac{1}{n}\log \mathbb{P}[|\mathcal{C}_{o_A}| = n] \geq -\frac{1}{n}\log \mathbb{P}[|\mathcal{C}_{o_B}| = n] + \frac{\delta}{2}.$$

By rewriting it follows that

$$\mathbb{P}[|\mathcal{C}_{o_A}| = n] \leq e^{-\frac{\delta}{2}n}\mathbb{P}[|\mathcal{C}_{o_B}| = n]. \tag{4.14}$$

It holds that $\mathbb{P}[|\mathcal{C}_{o_A}| = n] \geq \mathbb{P}[|\mathcal{C}_{o_A}| = n, C_{o_A} \cap \mathbb{X}_B \neq \varnothing]$. By Corollary 4.15 we find that

$$\mathbb{P}[|\mathcal{C}_{o_A}| = n, C_{o_A} \cap \mathbb{X}_B \neq \varnothing] = \frac{\rho(B)}{\rho(A)} \frac{\mathbb{E}[|\mathcal{C}_{o_B} \cap \mathbb{X}_A| \mid |\mathcal{C}_{o_B}| = n]}{\mathbb{E}[|\mathcal{C}_{o_A} \cap \mathbb{X}_B| \mid |\mathcal{C}_{o_A}| = n, \mathcal{C}_{o_A} \cap \mathbb{X}_B \neq \varnothing]} \mathbb{P}[|\mathcal{C}_{o_B}| = n].$$

We can now upper bound the denominator by

$$\mathbb{E}[|\mathcal{C}_{o_A} \cap \mathbb{X}_B| \mid |\mathcal{C}_{o_A}| = n, \mathcal{C}_{o_A} \cap \mathbb{X}_B \neq \varnothing] \leq n.$$

By (**A1**) we know that for sufficiently large n it must holds that

$$\mathbb{E}[|\mathcal{C}_{o_B} \cap \mathbb{X}_A| \mid |\mathcal{C}_{o_B}| = n] \geq c,$$

for some $c > 0$.

Combining we find that

$$\mathbb{P}[|\mathcal{C}_{o_A}| = n, C_{o_A} \cap \mathbb{X}_B \neq \varnothing] \geq \frac{\rho(B)}{\rho(A)} \frac{c}{n} \mathbb{P}[|\mathcal{C}_{o_B}| = n].$$

This is a contradiction to (4.14), for sufficiently large n. Note that we require assumption (**A1**) for this to hold true, as it guarantees that $\mathbb{P}[\mathcal{C}_{o_B} \cap \mathbb{X}_A \neq \varnothing] \neq 0$. And thus $\zeta^A = \zeta^B$. ■

4.5.3 Proof of Theorem 4.4

The following proof closely follows the proof on the log bound of the Poisson Boolean model by Penrose in [Pen03, Theorem 10.3].

Proof of 4.4. We start by showing that the largest component in a box with side lengths $2s$ is no larger than $\frac{d}{\zeta(\lambda)} \log 2s$. Let $\alpha > \frac{d}{\zeta(\lambda)}$. By applying the Markov bound and then the Mecke formula we find

$$\begin{aligned}
\mathbb{P}[|L_1(\xi_\lambda \cap \Lambda_s)| \geq \alpha \log 2s] &\leq \mathbb{E}\left[\sum_{x \in \eta} \mathbf{1}\{|\mathcal{C}(x, \xi^x \cap \Lambda_s)| \geq \alpha \log 2s\}\right] \\
&= \lambda \int_{\Lambda_s} \int_{\mathbb{M}} \mathbb{P}[|\mathcal{C}(x_m, \xi^{x_m} \cap \Lambda_s)| \geq \alpha \log 2s] \rho(\mathrm{d}m) \mathrm{d}x
\end{aligned}$$

Now consider some $\zeta' \in (\frac{d}{\alpha}, \zeta(\lambda))$. By definition (**C3**) of $\zeta(\lambda)$ as a limit, we know that for s large enough

$$\begin{aligned}
\int_{\mathbb{M}} \mathbb{P}[|\mathcal{C}(x_m, \xi^{x_m} \cap \Lambda_s)| \geq \alpha \log 2s \text{ in } \xi_\lambda^{x_m}] \rho(\mathrm{d}m) &\leq \int_{\mathbb{M}} \mathbb{P}[|\mathcal{C}_{o_m}| \geq \alpha \log 2s \text{ in } \xi_\lambda^{o_m}] \rho(\mathrm{d}m) \\
&\leq \exp(-\zeta' \alpha \log 2s) = (2s)^{-\zeta' \alpha}.
\end{aligned}$$

Hence, by our choices of α and ζ', we find

$$\mathbb{P}[|L_1(\xi_\lambda \cap \Lambda_s)| \geq \alpha \log 2s] \leq \lambda (2s)^{d - \zeta' \alpha} \xrightarrow{s \to \infty} 0.$$

For the other direction we choose $\beta < d/\zeta(\lambda)$ and $\zeta'' \in (\zeta(\lambda), d/\beta)$. We will tile the box Λ_s with smaller boxes. Let $m(s) = \lfloor \frac{s}{\beta \log 2s} \rfloor^d$ denote the number of boxes. Let $\{A_{1,s}, \ldots, A_{m(s),s}\}$ be the maximal collection of disjoint boxes with side-length $2s/\sqrt[d]{m(s)} \geq 2\beta \log 2s$, where, by our choice of side-length, the $A_{i,s}$ fill Λ_s exactly. Let $x_{i,s}$ denote the center of the box $A_{i,s}$. Now consider $\lambda' \in (0, \lambda)$ such that $\zeta(\lambda') < d/\beta$.

This is possible by the continuity of ζ. We can separate η_λ into a union of $\eta_{\lambda'}$ and

$$\eta_{\lambda',\lambda} = \big\{(X_i, \lambda_i) \bigm| \lambda' \le \lambda_i < \lambda\big\}.$$

Now take $\zeta(\lambda') < \zeta'' < d/\beta$.

Let $x \in \mathbb{R}^d$ and $r > 0$. Denote by $B_r(x)$ the closed ball with radius r centered at x. If $\eta_{\lambda',\lambda} \cap B_1(x_{i,s})$ consists of a single point we denote that point by $X_{i,s}$ and let

$$V_{i,s} := |\mathcal{C}(X_{i,s}, \xi_{\lambda'}^{X_{i,s}} \cap B_{i,s})|,$$

where $X_{i,s}$ inherits the connections from the original sampling of ξ_λ. If $|\eta_{\lambda',\lambda} \cap B_{i,s}| \neq 1$ then let $V_{i,s} = 0$. Let μ be the volume of a d-dimensional unit ball. By our choice of box size we know that $\{0 < V_{i,s} < \beta \log 2s\} \subset \{\mathcal{C}_{X_{i,s}}(\xi_{\lambda'}) \subset B_{i,s}\}$ and so $V_{i,s}$ has the distribution of the size of the component of the origin. Then, by independence of $\eta_{\lambda'}$ and $\eta_{\lambda',\lambda}$, we find for large s that

$$\begin{aligned}\mathbb{P}[V_{i,s} \ge \beta \log s] &\ge \mu(\lambda - \lambda')e^{-\mu(\lambda-\lambda')} \int_{\mathbb{M}} \mathbb{P}[|\mathcal{C}_{o_m}| \ge \beta \log 2s \text{ in } \xi_{\lambda'}^{o_m}]\rho(\mathrm{d}m) \\ &\ge c' \exp(-\zeta'' \beta \log 2s) = c' s^{-\zeta''\beta},\end{aligned}$$

where the inequality follows from the definition of ζ and $c' = \mu(\lambda - \lambda')e^{-\mu(\lambda-\lambda')}$. The random variables $V_{i,s}$ are independent, since they are dependent on configurations in disjoint boxes. It follows that

$$\begin{aligned}\mathbb{P}\Big[\bigcap_{i=1}^{m(s)} \{V_{i,s} < \beta \log 2s\}\Big] &\le \big(1 - c' s^{-\zeta''\beta}\big)^{m(s)} \\ &\le \exp(-c' s^{-\zeta''\beta} m(s))\end{aligned}$$

which tends to zero by the definition of $m(s)$ and the fact that $\zeta''\beta < d$. On the other hand, if for some i we have $V_{i,s} \ge \beta \log 2s$, then $L_1(\xi_\lambda \cap \Lambda_s) \ge \beta \log 2s$. This gives us the desired result. ■

4.6 Large Poisson Deviations

Theorem 4.22. *Let $c > 0$ and $a > \lambda > 0$. Then for any $c' \in (0, a \log(\frac{a}{\lambda}) + \lambda - a)$ and all sufficiently large $k > 0$ we find that*

$$\mathbb{P}[\mathrm{Pois}(\lambda k + c) \ge ak] \le \exp(-c' k).$$

See also [Pen03].

Proof. We use the Chernoff bound, which is a direct consequence of the Markov inequality. Let X be a random variable with a well defined moment generating function and $a \in \mathbb{R}$. Then

$$\mathbb{P}[X \geq a] = \mathbb{P}[e^{tX} \geq e^{ta}] \leq e^{-ta}\mathbb{E}[e^{tX}].$$

This bound only holds if $t > 0$, since $x \mapsto e^{tx}$ is only increasing for such t. We can take the infimum with respect to all such t. This gives us

$$\mathbb{P}[X \geq a] \leq \inf_{t>0} e^{-ta}\mathbb{E}[e^{tX}].$$

We can now apply this to a Poisson random variable. Let $X \sim \mathrm{Pois}(\lambda)$ and $a \geq 0$. Using the fact that $\mathbb{E}[e^{tX}] = e^{\lambda(e^t-1)}$ we obtain

$$\mathbb{P}[X \geq a] \leq \inf_{t\in\mathbb{R}} e^{-ta+\lambda e^t-\lambda}.$$

To find the infimum we take the derivative:

$$\frac{d}{dt}e^{-ta+\lambda e^t-\lambda} = (-a+\lambda e^t)\underbrace{e^{-ta+\lambda e^t-\lambda}}_{>0} \stackrel{!}{=} 0.$$

By solving for 0 we obtain the following:

$$t^* = \log(\frac{a}{\lambda}).$$

Note that by our choice of t that this bound only holds when $a > \lambda$. Plugging back into our original bound we finally find

$$\mathbb{P}[X \geq a] \leq e^{-\log(a/\lambda)a+a-\lambda}.$$

Now we can use the values $\lambda k + c$ and ak where $a > \lambda > 0$ and $c \in \mathbb{R}$. Then if $k > c/(a-\lambda)$ we find that

$$\begin{aligned}\mathbb{P}[\mathrm{Pois}(\lambda k + c) \geq ak] &\leq \exp\big(-k\underbrace{(a\log(\frac{ak}{\lambda k + c}) + \lambda - a + \frac{c}{k})}_{>0}\big)\\ &\leq e^{-c'k}.\end{aligned}$$

■

Chapter 5

The Supercritical Regime

Percolation models behave in significantly different ways when they are supercritical, which in our case means $\lambda > \lambda_c$. The key difference is that the model now contains an infinite component. In essence, the study of the supercritical regime reduces to understanding the behavior of the infinite component. For us the local connectivity of the infinite component will be most important.

Large parts of this chapter concern themselves with adopting ideas from [CMT24] to the MRCM. The results are needed to ensure that long paths connect with high probability within a bounded region. The challenges of translating the methods are more apparent in this chapter than the previous, and in particular some statements are outright false in this setting.

As in the previous chapter, the Stopping Set lemma will be a key tool.

5.1 Statement

The key statement we want to prove is that in the super-critical regime the largest component takes up a $\theta(\lambda)$ proportion of all points in a box. This is a natural conjecture as we might expect the largest component in Λ_s to approximately be the intersection of the infinite component with Λ_s. Note that by Mecke

$$\mathbb{E}[|\mathcal{C}^\infty \cap \Lambda_s|] = \mathbb{E}\left[\sum_{x\in\eta\cap\Lambda_s} \mathbf{1}\{x \leftrightarrow \infty\}\right] = \lambda(2s)^d\theta(\lambda).$$

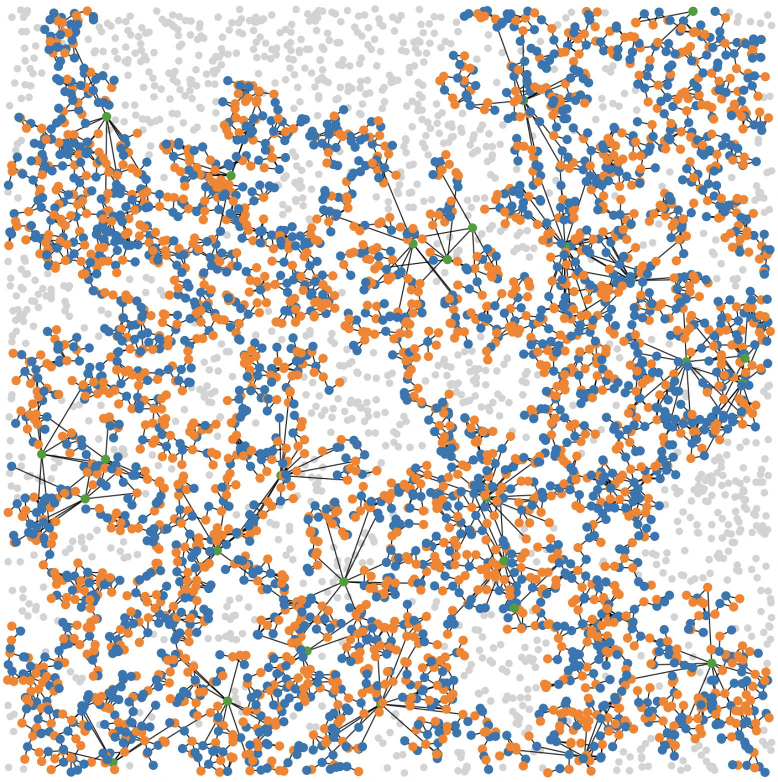

Figure 5-1: An instance of the MRCM at a supercritical intensity. The largest component is highlighted.

The key difficulty lies in proving that the infinite component doesn't get cut in unlikely ways by the box, thereby dividing $\mathcal{C}^\infty$ into smaller components inside the box. The following theorem essentially states that this does not happen.

Theorem 5.1. *Assuming* (**A2**) *(defined below),* $d \geq 2$ *and* $\lambda > \lambda_c(\psi)$ *we have that*

$$\lim_{s \to \infty} \frac{\mathbb{E}[|L_1(\Lambda_s)|]}{\lambda(2s)^d} = \theta(\lambda, \psi).$$

For intuition, notice in Figure 5-1 that the fraction of points that are occupied by the largest component are not evenly spread out, but rather clump together and leave holes. To deal with this difficulty we will employ a coarse-graining argument that will allow us to work at a scale where we may essentially ignore such defects.

5.1.1 Outline

In Section 5.2 we prove an upper bound on the 'two-arm event', the event where an inserted point connects to the complement of a ball with two disjoint paths, following ideas from [CMT23]. In Section 5.3 we expand the previous bound to get a lower bound on the probability on the event that two disjoint components leave an annulus.

We use this bound to prove a classic result by Grimmett and Marstrand in Chapter 6, following ideas from [DKT21]. It states that in the supercritical regime we can find a sufficiently thick two-dimensional slab such that the process also percolates in this slab. We then use the Grimmett-Marstrand result to improve our uniqueness bounds in Section 6.2.

Finally, in Section 6.3, we use the previous bounds to 'glue' paths together and prove Theorem 5.1 following ideas from [Pen22].

5.1.2 Assumptions

We will need to assume the following lemma as it will be out of the scope of this thesis to prove.

Lemma 5.2. *Let $\lambda > \lambda_c$. There exists some $c, \delta > 0$ such that for all $s \geq 1$ we have:*

$$\mathbb{P}[B_s \leftrightarrow \infty] \geq 1 - cs^{-\delta}. \tag{\textbf{A2}}$$

Note that once we have proven Theorem 5.1 (using (**A2**)) the above assumption holds with $\delta = d$. Moreover, we will prove a result in Chapter 6 which will allow us to show that the decay must be exponential (assuming (**A2**)). Hence, this is a fairly modest assumption. The difficulty lies in proving this statement ad hoc. We will require (**A2**) to ensure the existence of certain long paths. Together with 'uniqueness' statements which we will prove in the following two sections this will be the key tool to construct large connected components.

In lattice percolation (**A2**) has been shown without further assumptions, see e.g. [CMT23]. They rely on a bound by Talagrand [Tal94] that does not, in our case, hold for the Poisson point processes. A version of this bound has been proven in [NPY19], but requires additional assumptions that do not hold in our case. In particular, they provide a counterexample to the general statement. In essence the issue with Poisson point processes is that an arbitrarily large number of points may clump together.

5.1.3 Preliminary Results

We will require the following two facts about the Poisson distribution before we can continue. The first lemma states that Poissonian 'overshoots' are in expectation not more than their expected value.

Lemma 5.3. *Let $X \sim \mathrm{Pois}(\lambda)$ where $\lambda \in \mathbb{R}_{>}$, and let $a \in \mathbb{N}$. Then*

$$\mathbb{E}[X - a \mid X \geq a] \leq \lambda.$$

Proof.

$$\mathbb{E}[X\mathbf{1}\{X \geq a\}] = \sum_{k \geq a} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda \mathbb{P}[X \geq a-1].$$

By the definition of conditional expectation we have

$$\mathbb{E}[X \mid X \geq a] = \lambda \frac{\mathbb{P}[X \geq a-1]}{\mathbb{P}[X \geq a]} = \lambda \left(1 + \frac{\mathbb{P}[X = a-1]}{\mathbb{P}[X \geq a]}\right). \tag{5.1}$$

Then we bound

$$\lambda \frac{\mathbb{P}[X = a-1]}{\mathbb{P}[X \geq a]} = \lambda \frac{\lambda^{a-1}/(a-1)!}{\sum_{k \geq a} \frac{\lambda^k}{k!}} = a \frac{\lambda^a/a!}{\sum_{k \geq a} \frac{\lambda^k}{k!}} \leq a. \tag{5.2}$$

Combining (5.1) and (5.2) yields: $\mathbb{E}[X \mid X \geq a] \leq \lambda + a$. ∎

The following bound on the moments of Poisson random variables is from [Ahl22] (where the author proves a sharper and more general bound).

Lemma 5.4. *Let $X \sim \mathrm{Pois}(\lambda)$ and $k \geq 0$, then*

$$\mathbb{E}[X^k] \leq \lambda^k \exp\left(\frac{k^2}{2\lambda}\right).$$

5.2 Two-arm Bound

The goal of this Section is to prove an upper bound on the 'two-arm' event. In words the two-arm event determines if two disjoint clusters are close enough to each other that a single added point can connect them.

Let K be a compact subset of $\mathbb{R}^d$ and $x \in K \times \mathbb{M}$. Denote by $\mathfrak{C}_K := \mathfrak{C}_K(\xi_\lambda)$ the collection of connected components in $\xi_\lambda \cap K$ that are connected to K^c in ξ_λ. For each component we only consider the points that are in K. Furthermore, we consider two such components separate even if they connect outside K. We denote by $\mathfrak{C}_K^x$ the collection of such components in the graph ξ_λ^x. Consider the following event

$$Arm_x(K) := \{|\mathfrak{C}_K^x| < |\mathfrak{C}_K|\}.$$

In words, the arm event happens when x connects to K^c through two (or more) components that are only connected through x. We will write $Arm_x(t) := Arm_x(B_t)$ and $Arm(t) := Arm_o(t)$.

For subsets $K \subseteq \mathbb{R}^d$ we introduce the following notation. First, we write $|K|$ to denote the Lebesgue measure of K. We also define the set $\partial^{\text{in}} K := \{x \in K \mid d(x, K^c) \leq 1\}$.

The goal of this section is bound the probability of the two-arm event as a function of t. We will require the following technical lemma adapted from [CMT23] to the continuum.

Lemma 5.5. *Let $K \subset \mathbb{R}^d$ be a measurable compact set. Let $\varepsilon \in (0, \frac{1}{2})$. Let h be a function from connected components to the real numbers. Then*

$$\mathbb{E}\left[\sum_{C \in \mathfrak{C}_K} h(C)\right] \leq \lambda \bigg(\int_K \int_{\mathbb{M}} \mathbb{E}\left[\frac{h(\mathcal{C}_{x_m})^2}{|\mathcal{C}_{x_m}|^{1+\varepsilon}}\right] \rho(\mathrm{d}m)\mathrm{d}x \bigg)^{\frac{1}{2}} \Big(|K|^{\varepsilon} |\partial^{in} K|^{1-\varepsilon} \Big)^{\frac{1}{2}}.$$

Lemma 5.5 might seem unnatural at first, but it will help us reduce the proof of Proposition 5.6 to finding a suitable function h.

Proof. For this proof we will write $\mathfrak{C} = \mathfrak{C}_K$ and drop explicit mentions of marks. First, we notice that we can replace the sum over components by a sum over points in those components as long as we divide by the number of points in that given component. By the Mecke formula

$$\begin{aligned}\mathbb{E}\left[\sum_{C \in \mathfrak{C}} h(C)\right] &= \mathbb{E}\left[\sum_{x \in \eta \cap K} \frac{h(\mathcal{C}_x)}{|\mathcal{C}_x|} \mathbf{1}\{x \leftrightarrow K^c\}\right] \\ &= \lambda \int_K \mathbb{E}\left[\frac{h(\mathcal{C}_x)}{|\mathcal{C}_x|} \mathbf{1}\{x \leftrightarrow K^c\}\right] \mathrm{d}x\end{aligned}$$

We require the indicator $\mathbf{1}\{x \leftrightarrow K^c\}$ to ensure that $\mathcal{C}_x \in \mathfrak{C}$. Next we apply the Cauchy-Schwarz inequality. We use the fact that $\int \mathbb{E}[f(x)g(x)]\mathrm{d}x$ is an inner product.

We choose $f(x) = \frac{h(\mathcal{C}_x)}{|\mathcal{C}_x|^{(1+\varepsilon)/2}}$ and $g(x) = \frac{\mathbf{1}\{x \leftrightarrow K^c\}}{|\mathcal{C}_x|^{(1-\varepsilon)/2}}$. We find that

$$\mathbb{E}\left[\sum_{C\in\mathfrak{C}} h(C)\right] \leq \lambda \left(\int_K \mathbb{E}\left[\frac{h(\mathcal{C}_x)^2}{|\mathcal{C}_x|^{1+\varepsilon}}\right] dx\right)^{\frac{1}{2}} \left(\int_B \mathbb{E}\left[\frac{\mathbf{1}\{x \leftrightarrow K^c\}}{|\mathcal{C}_x|^{1-\varepsilon}}\right] dx\right)^{\frac{1}{2}}.$$

The contents of the left parentheses are already as desired. Now we have to simplify the contents of the right parentheses. We use the Mecke equation again (in the other direction) together with the fact that the sum over all points reaching K^c is equal to the sum over all components reaching K^c multiplied by their number of vertices. This gives

$$\begin{aligned}\int_K \mathbb{E}\left[\frac{\mathbf{1}\{x \leftrightarrow K^c\}}{|\mathcal{C}_x|^{1-\varepsilon}}\right] dx &= \frac{1}{\lambda}\mathbb{E}\left[\sum_{x\in\eta\cap K} \frac{\mathbf{1}\{x \leftrightarrow K^c\}}{|\mathcal{C}_x|^{1-\varepsilon}}\right] \\ &= \frac{1}{\lambda}\mathbb{E}\left[\sum_{C\in\mathfrak{C}} |C|^{\varepsilon}\cdot 1\right].\end{aligned}$$

Next, we apply Hölder's inequality to the sum inside the expected value with parameters $\frac{1}{\varepsilon}$ and $\frac{1}{1-\varepsilon}$, and again with the same parameters in the third inequality. Then,

$$\begin{aligned}\frac{1}{\lambda}\mathbb{E}\left[\sum_{C\in\mathfrak{C}} |C|^{\varepsilon}\cdot 1\right] &\leq \frac{1}{\lambda}\mathbb{E}\left[(\sum_{C\in\mathfrak{C}} |C|)^{\varepsilon}\cdot |\mathfrak{C}|^{1-\varepsilon}\right] \\ &\leq \frac{1}{\lambda}\mathbb{E}\left[(\sum_{x\in\eta\cap K} 1)^{\varepsilon}(\sum_{x\in\eta\cap\partial^{\text{in}}K} 1)^{1-\varepsilon}\right] \\ &\leq \frac{1}{\lambda}\mathbb{E}\left[\eta(K\times\mathbb{M})\right]^{\varepsilon}\mathbb{E}\left[\eta(\partial^{\text{in}}K\times\mathbb{M})\right]^{1-\varepsilon} \\ &= |K|^{\varepsilon}|\partial^{\text{in}}K|^{1-\varepsilon}.\end{aligned}$$

To get to the second line from the first line note that the total number of points in the components of $\mathfrak{C}$ is naturally upper bounded by the number of vertices in K. Similarly, the number of components $|\mathfrak{C}|$ is bounded by the number of vertices in the boundary $\partial^{\text{in}}K$, since each component has to reach K^c by definition. ■

We will use Lemma 5.5 to show the following key result.

Proposition 5.6. *Let $\lambda > 0$. There exists some $\varepsilon_0 > 0$, $t_0 > 0$ and $c_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $t > t_0$ we have*

$$\mathbb{P}_\lambda[Arm(t)] \leq \frac{c_0}{\sqrt{\varepsilon}} t^{-\frac{1}{2}+\varepsilon}.$$

Note that the mark of the origin is sampled randomly by $\mathbb{P}$.

By setting $\varepsilon = \frac{1}{\log t}$ and noting that $t^{1/\log t} = e$ we get the following corollary.

Corollary 5.7. *Let $\lambda > 0$ and $t > e$ we have*

$$\mathbb{P}_\lambda[Arm(t)] \leq c \left(\frac{\log(t)}{t}\right)^{\frac{1}{2}},$$

where $c = ec_0$, c_0 is as in Proposition 5.6 and e is Euler's number.

We remind the reader of the following notation. If $A \subset \mathbb{X}$ is a locally finite set, we define

$$\psi^A(x) := 1 - \prod_{y \in A}(1 - \psi(x, y)), \tag{2.2}$$

to be the probability of x connecting to at least one vertex in A. The following proof adapts an argument in [CMT23] to the MRCM.

Proof of Proposition 5.6. We fix $t > 2$ and $s = (t+1)/2$. We notice that for all $x \in B_{s-1}$

$$\mathbb{P}[Arm_o(t)] \leq \mathbb{P}[Arm_o(B_s(x))] = \mathbb{P}[Arm_{-x}(B_s)],$$

since $Arm_o(t)$ implies $Arm_o(B_s(x))$. Note that there is a 'buffer' of distance 1 between $x \in B_{s-1}$ and B_s to guarantee that x can not share an edge with a point outside B_s, otherwise the above implication could break. Integrating x over B_{s-1} we find

$$\begin{aligned}\mathbb{P}[Arm(t)] &\leq \frac{1}{|B_{s-1}|}\int_{B_{s-1}} \mathbb{P}[Arm_x(s)]\mathrm{d}x \\ &\leq \frac{1}{|B_{s-1}|}\int_{B_s} \mathbb{P}[Arm_x(s)]\mathrm{d}x.\end{aligned} \tag{5.3}$$

We want to apply Lemma 5.5, which means we have to find the right h to upper-bound $\mathbb{E}\big[\sum_{x\in\eta\cap B_s} \mathbf{1}\{Arm_x(s)\}\big]$. We write $\mathfrak{C} := \mathfrak{C}_{B_s}$ and $\mathfrak{C}^x := \mathfrak{C}^x_{B_s}$. We define $\overline{C} := \bigcup_{C\in\mathfrak{C}} C$ to be the collection of all points that are connected to B_s^c. Similarly, we

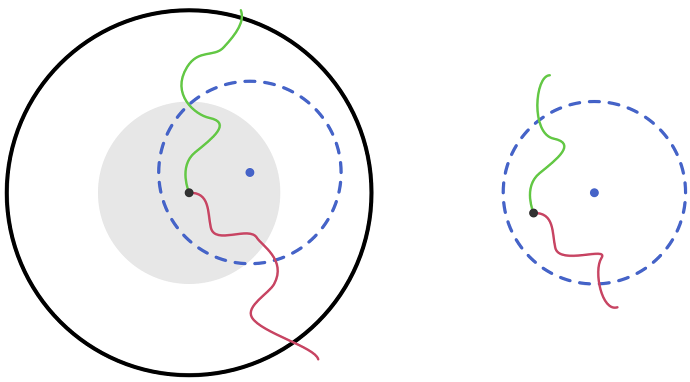

Figure 5-2: On the left hand side we sketch out $Arm_o(t)$, with a point x and $B_s(x)$ overlaid in blue. On the right hand side we sketch $Arm_o(B_s(x))$, re-centered on x. Notice that $Arm_o(t)$ necessarily implies $Arm_o(B_s(x))$.

define $\overline{C}^x := \bigcup_{C\in\mathfrak{C}^x} C$. We observe that

$$\begin{aligned}\int_{B_s} \mathbb{P}[x \sim \overline{C}]\mathrm{d}x &= \int_{B_s} \mathbb{P}\left[x \leftrightarrow B_s^c\right]\mathrm{d}x - \int_{\partial^{\text{in}} B_s} \mathbb{P}\left[x \leftrightarrow B_s^c, x \not\sim \overline{C}\right]\mathrm{d}x \\ &\geq \frac{1}{\lambda}\mathbb{E}\left[\sum_{x\in\eta\cap B_s} \mathbf{1}\{x \leftrightarrow B_s^c\}\right] - \lambda c' s^{d-1} \\ &= \frac{1}{\lambda}\mathbb{E}\left[\sum_{C\in\mathfrak{C}} |C|\right] - \lambda c' s^{d-1},\end{aligned} \tag{5.4}$$

for some constant $c' > 0$.

For a given point x, we have that the event $Arm_x(s)$ occurs if x connects to two (or more) $C \in \mathfrak{C}$. Explicitly $Arm_x(s) = \{\sum_{C\in\mathfrak{C}} \mathbf{1}\{x \sim C\} \geq 2\}$. We also know that for any random variable X taking values in $\mathbb{N}$ it holds that $\mathbb{P}[X \geq 2] \leq E[X] - \mathbb{P}[X \geq 1]$[1]. Using (5.4) in line 2 we find that

$$\begin{aligned}\int_{B_{s-1}} \mathbb{P}[Arm_x(s)]\mathrm{d}x &\leq \int_{B_{s-1}} \mathbb{E}\Bigg[\left(\sum_{C\in\mathfrak{C}} \mathbf{1}\{x \sim C\}\right) - \mathbf{1}\{x \sim \overline{C}\}\Bigg]\mathrm{d}x \\ &\leq \mathbb{E}\left[\sum_{C\in\mathfrak{C}} \left(\int_{B_s\times\mathbb{M}} \psi^C(x)\mathrm{d}x - \frac{1}{\lambda}|C|\right)\right].\end{aligned} \tag{5.5}$$

[1]Using the fact that $\mathbb{E}[X] = \sum_{n\in\mathbb{N}} \mathbb{P}[X \geq n]$.

We now want to use Lemma 5.5 with

$$h(C) = \lambda \int_{B_s \times \mathbb{M}} \psi^C(x)\mathrm{d}x - |C|. \tag{5.6}$$

To prepare for the final step we first show

$$\mathbb{E}\left[\frac{h(\mathcal{C}_x)^2}{|\mathcal{C}_x|^{1+\varepsilon}}\right] \leq c_\varepsilon < \infty, \tag{5.7}$$

where c_ε is independent of s and x. The strategy will be to construct $\mathcal{C}_x$ iteratively in the same manner as in the proof of the Stopping Set lemma, except we stop the exploration outside of B_s. We start with $A_{-1} = \varnothing$ and $A_0 = \{x\}$. We construct A_i by adding all neighbors of A_{i-1} in B_s. For simplicity of notation we define the increment $N_{t+1} := A_{t+1} \setminus A_t$. We also define the following integrals

$$\alpha_t := \lambda \int_{B_s \times \mathbb{M}} \psi^{A_{t-1}}(x)\mathrm{d}x \quad \text{ and } \quad \beta_t := \lambda \int_{B_s \times \mathbb{M}} \psi^{N_{t-1}}(x)(1 - \psi^{A_{t-2}}(x))\mathrm{d}x.$$

Note that $\alpha_0 = 0$

To bound $h(\mathcal{C}_x)$ we will construct a martingale X_t whose terminal value is $h(\mathcal{C}_x)$. Consider the filtration $\mathcal{F}_t := \sigma(A_0, \dots, A_t)$. We will use the following three facts that arise in the proof of the Stopping Set lemma.

1. The sets $\eta \setminus A_{t+1}$ and $A_{t+1} \setminus A_t$ are independent given $\mathcal{F}_t$.
2. The set $\eta \setminus A_{t+1}$ is distributed like a Poisson point process with intensity $\lambda(1 - \psi^{A_t})$ given $\mathcal{F}_t$.
3. Given $\mathcal{F}_t$ the set N_{t+1} is distributed like a Poisson point process with intensity $\lambda\psi^{N_t}(1 - \psi^{A_{t-1}})$.

We define $X_t := |A_t| - \alpha_t$. We will show that this is a martingale with the needed properties.[2] The identity

$$\begin{aligned} \psi^{A_t} - \psi^{A_{t-1}} &= 1 - (1 - \psi^{N_t})(1 - \psi^{A_{t-1}}) - \psi^{A_{t-1}} \\ &= \psi^{N_t}(1 - \psi^{A_{t-1}}) \end{aligned} \qquad \forall t \geq 0,$$

shows that $\alpha_t - \alpha_{t-1} = \beta_t$ and so $\alpha_t = \sum_{s \leq t} \beta_s$. We verify that $(X_t)_{t\geq 0}$ is indeed a

[2] X_t is defined as the negation of h, as this is the more natural choice from a martingale perspective. This will not matter as we will square h later.

martingale using fact 3. from above:

$$\begin{aligned}\mathbb{E}[X_{t+1} - X_t \mid \mathcal{F}_t] &= \mathbb{E}[|N_{t+1}| - \lambda \int_{B_s \times \mathbb{M}} \psi^{N_t}(1 - \psi^{A_{t-1}}) \mathrm{d}y \mid \mathcal{F}_t] \\ &= \mathbb{E}\big[|N_{t+1}| - \mathbb{E}[|N_{t+1}| \mid \mathcal{F}_t] \mid \mathcal{F}_t\big] = 0.\end{aligned}$$

Note that $X_0 = 1$. Let $T := \inf\{t \in \mathbb{N} \mid N_t = \varnothing\}$ be the almost surely finite random time when no new points in B_s are path-connected to x. Notice that by definition of T and the exploration process we have $A_T = A_{T-1} = \mathcal{C}_x$. Similarly, we also find that $X_T = X_{T-1}$. Thus,

$$X_T = |A_T| - \lambda \int_{B_s \times \mathbb{M}} \psi^{A_T} \mathrm{d}x = -h(\mathcal{C}_x).$$

Note that $(\alpha_t)_t$ and $(\beta_t)_t$ are predictable sequences: they are measurable with respect to $\mathcal{F}_{t-1}$. Then, using the fact that $|A_t \setminus A_{t-1}| \sim \mathrm{Pois}(\beta_t)$ conditional on $\mathcal{F}_{t-1}$ we find

$$\begin{aligned}\mathbb{E}[(X_t - X_{t-1})^2] &= \mathbb{E}[(|N_t| - \beta_t)^2] \\ &= \mathbb{E}\big[\mathbb{E}[\beta_t^2 - 2\beta_t|N_t| + |N_t|^2 \mid \mathcal{F}_{t-1}]\big] \\ &= \mathbb{E}\big[\mathbb{E}[\beta_t^2 - 2\beta_t^2 + \beta_t^2 + \beta_t \mid \mathcal{F}_{t-1}]\big] \\ &= \mathbb{E}[|A_t \setminus A_{t-1}|].\end{aligned}$$

We know by orthogonality of martingale increments that

$$\mathbb{E}[X_t^2] - \mathbb{E}[X_0^2] = \sum_{i=1}^{t} \mathbb{E}[(X_i - X_{i-1})^2] = \sum_{i=1}^{t} \mathbb{E}[|A_i \setminus A_{i-1}|] = \mathbb{E}[|A_t|] - 1. \tag{5.8}$$

This can be interpreted as saying that the variance of the exploration process is exactly the size of the component at that point in the process, as might be expected from a Poisson point process.

Next we will require a bound on $\mathbb{E}[X_T^2 \mathbf{1}_{\{|A_T| \in [a,b]\}}]$ for any $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$. We introduce $S_b := \min\{t \geq 0 : |A_t| > b\}$ to be the stopping time where $|A_t|$ first exceeds b. It is possible for the event $\{S_b = \infty\}$ to have positive probability, i.e. if $|A_T| \leq b$. On the other hand, if $S_b < \infty$ it immediately follows that $S_b < T$, since T is the last time A_t can increase. We now drop the subscript $S = S_b$ for notational convenience. Notice that:

$$\mathbb{E}\left[X_T^2 \mathbf{1}_{\{|A_T| \in [a,b]\}}\right] \leq \mathbb{E}\left[X_T^2 \mathbf{1}_{\{|A_T| \leq b\}}\right] = \mathbb{E}\left[X_{T \wedge S}^2 \mathbf{1}_{\{S = \infty\}}\right]. \tag{5.9}$$

The stopping time $T \wedge S$ is almost surely at most b, since the slowest $|A_t|$ can increase without stopping is 1 every time step. If $|A_t|$ stops increasing any time before reaching b then $T \leq b$. By the Optional Stopping theorem (applied to the martingale $X_t^2 - |A_t|$) and (5.8) we can write $0 = \mathbb{E}[X_{T\wedge S}^2 - |A_{T\wedge S}|]$. We multiply by $1 = \mathbf{1}_{\{S=\infty\}} + \mathbf{1}_{\{S<\infty\}}$ and rewrite as

$$\mathbb{E}\left[X_{T\wedge S}^2 \mathbf{1}_{\{S=\infty\}}\right] = \mathbb{E}\left[|A_{T\wedge S}|\mathbf{1}_{\{S=\infty\}} + |A_{T\wedge S}|\mathbf{1}_{\{S<\infty\}} - X_S^2 \mathbf{1}_{\{S<\infty\}}\right]. \tag{5.10}$$

We can bound $|A_{T\wedge S}|\mathbf{1}_{\{S=\infty\}} \leq b$ and $-X_S^2\mathbf{1}_{\{S<\infty\}} \leq 0$. Next we bound $|A_{T\wedge S}|\mathbf{1}_{\{S<\infty\}}$. Since the increment $|N_S|$ has to be strictly larger than $b - |A_{S-1}|$, and by definition of S we get

$$|N_S| > b - |A_{S-1}| \geq 0.$$

Next we wish to apply Lemma 5.3 to $|N_S|$. But since S is a stopping time $|N_S|$ need not be Poisson distributed. We want to say that "$|N_S| \sim \mathrm{Pois}(\beta_S)$". To make this rigorous we condition on $\{S = k\} = \{|N_k| > b - |A_{k-1}|\} \cap \{|A_{k-1}| \leq b\}$.

$$\begin{aligned}
\mathbb{E}[|A_{T\wedge S}|\mathbf{1}_{S<\infty}] &= \sum_{k\geq 1} \mathbb{E}[|A_k|\mathbf{1}\{S=k\}] \\
&= \sum_{k\geq 1} \mathbb{E}\left[\mathbb{E}\left[(|N_k| + |A_{k-1}|)\mathbf{1}\{|N_k| > b - |A_{k-1}|\} \mid \mathcal{F}_{k-1}\right] \mathbf{1}\{|A_{k-1}| \leq b\}\right]
\end{aligned}$$

Now using Lemma 5.3 (rewritten as $\mathbb{E}[X\mathbf{1}\{X \geq a\}] \leq (\lambda + a)\mathbb{P}[X \geq a]$) we find

$$\mathbb{E}[|A_{T\wedge S}|\mathbf{1}_{S<\infty}] \leq \sum_{k\geq 1} \mathbb{E}\left[(\beta_k + b + 1)\mathbb{P}[S = k \mid \mathcal{F}_{k-1}]\mathbf{1}\{|A_{k-1}| \leq b\}\right].$$

By the union bound, conditionally on $|A_{k-1}| \leq b$, we have that $\beta_k \leq \lambda Z_\psi^\infty N_{k-1} \leq \lambda Z_\psi^\infty b$. Inserting this bound above we find:

$$\mathbb{E}[|A_{T\wedge S}|\mathbf{1}_{S<\infty}] \leq \sum_{k\geq 1}(Z_\psi^\infty + 2)b \cdot \mathbb{P}[S=k] \leq (\lambda Z_\psi^\infty + 2)b.$$

Combining the above display with (5.10) and (5.9) we get

$$\mathbb{E}\left[X_T^2 \mathbf{1}_{\{|A_T \in [a,b]|\}}\right] \leq (\lambda Z_\psi^\infty + 3)b.$$

We remind the reader of the definition of h (5.6). By using the above bound we can show (5.7):

$$\mathbb{E}\left[\frac{h(C_x)^2}{|C_x|^{1+\varepsilon}}\right] = \mathbb{E}\left[\frac{X_T^2}{|A_T|^{1+\varepsilon}}\right] \leq \sum_{i\geq 0} \frac{1}{2^{i(1+\varepsilon)}} \mathbb{E}\left[X_T^2 \mathbf{1}_{\{2^i \leq |A_T| \leq 2^{i+1}\}}\right]$$
$$\leq (Z_\psi^\infty + 3) \sum_{i\geq 0} \frac{2^{i+1}}{2^{i(1+\varepsilon)}} \leq 2\frac{\lambda Z_\psi^\infty + 3}{1-2^{-\varepsilon}}. \tag{5.11}$$

Now we will combine our efforts to get the bound we want. Combining (5.3) and (5.5) we get that

$$P[Arm(t)] \leq \frac{1}{\lambda |B_{s-1}|}\left(\mathbb{E}\left[\sum_{C\in\mathfrak{C}} h(C)\right] + \lambda c' s^{d-1}\right).$$

There exists some constant $c'' > 0$ such that $\frac{\lambda c' s^{d-1}}{\lambda |B_{s-1}|} \leq c'' s^{-1}$. We can now use Lemma 5.5 to find that

$$P[Arm(t)] \leq \left(\frac{1}{|B_{s-1}|}\int_{B_s} \mathbb{E}\left[\frac{h(\mathcal{C}_x)^2}{|\mathcal{C}_x|^{1+\varepsilon}}\right] \mathrm{d}x\right)^{\frac{1}{2}} \left(\frac{|\partial^{\text{in}} B_s|}{|B_{s-1}|}\right)^{\frac{1-\varepsilon}{2}} + c'' s^{-1}$$

Finally we apply (5.11). Notice that $\frac{1}{1-2^{-\varepsilon}} \leq 2\varepsilon^{-1}$ for $\varepsilon \in (0,1)$ and $\frac{|\partial^{\text{in}} B_s|}{|B_s|} \leq c_d t^{-1}$ for some constant c_d depending only on the dimension.[3] Thus, we can find some $c_\varepsilon = O(\varepsilon^{-\frac{1}{2}})$ such that the following holds

$$P[Arm(t)] \leq \left(2\frac{\lambda Z_\psi^\infty + 3}{1-2^{-\varepsilon}}\right)^{\frac{1}{2}} \left(\frac{|\partial^{\text{in}} B_s|}{|B_{s-1}|}\right)^{\frac{1-\varepsilon}{2}} + c'' s^{-1} \leq \frac{c_0}{\sqrt{\varepsilon}} t^{-\frac{1}{2}+\frac{\varepsilon}{2}}.$$

■

5.3 Uniqueness

It is our goal in Section 6.3 to construct long paths inside a box. In this section we will prove a bound on a certain uniqueness event which will allow us to 'glue' shorter paths together. We define the following event for $0 < r < s$:

$U(r,s) := U_\lambda(r,s) :=$

$\{$There is at most 1 cluster intersecting both B_r and B_s^c in $\xi \cap (B_{r-1}^c \cap B_{s+1})\}$.

Note that $U(r,s)$ allows there to be no crossing. One can interpret $U(r,s)^c$ as a stronger version of the two-arm event. We remind the reader of the notation $\tau_s(x,y)$ for the

[3]The bound follows from concavity of $1-2^{-x}$ on the interval $[0,1]$.

probability that $x \leftrightarrow y$ when only considering vertices in B_s (see Definition 2.2). The following has been adapted to this setting from [CMT23].

Proposition 5.8. *Assume $\lambda > 0$ such that $\theta(\lambda) > 0$. There exists some $\chi \in (0,1)$ and $c_2 > 0$ such that for all $r \geq 1$ sufficiently large it holds that*

$$\mathbb{P}[U_\lambda(h(r), r)] \geq 1 - c_2 r^{-1/4},$$

where $h(r) := \exp(\log(r)^\chi)$.

We remark that $h(r)$ grows faster than $\log(r)$, but slower than r^α for any $\alpha \in (0,1)$ as $r \to \infty$. We will denote the inverse of $h(r)$ by $H(r) = \exp(\log(r)^{1/\chi})$. Accordingly, $H(r)$ grows superpolynomially, but subexponentially as $r \to \infty$.

Recall the definition of $\overline{\tau}_\lambda(x, y)$, i.e. the restricted two-point function (2.4). It is equal to the probability that x connects to y via at least one vertex in η. Similarly, we define $\overline{\tau}_{\lambda,s}(x, y)$ as the restricted two-point on $\xi[\eta^{x,y} \cap B_s]$. By our assumptions on ψ it follows immediately that if $d(x, y) \geq 1$ then $\overline{\tau}_{\lambda,s}(x, y) = \tau_{\lambda,s}(x, y)$.

To prove the above Proposition 5.8 we require the following lemma which is adapted from [CMT24].

Lemma 5.9. *For all $\lambda > 0$, $\alpha \geq 1$ and $r, s, t \in \mathbb{R}$ such that $1 \leq r \leq s \leq t/2$ there exists some constant c_4 (depending on λ) such that*

$$\mathbb{P}[U_\lambda(r, t)^c] \leq c_4 \exp\left(\frac{2\alpha}{\lambda r^{d-1}}\right) \frac{r^{2d-2} s^d}{\inf_{x,y \in B_r} \overline{\tau}_s(x, y)} \mathbb{P}[Arm(t/2)]^{1-\frac{1}{\alpha}}.$$

Remark 5.10. Due to extra challenges which arise from the MRCM the bound in the above lemma is slightly weaker than the discrete equivalent in [CMT24]. Namely, the discrete version does not require the $1 - \frac{1}{\alpha}$ exponent or the related exp term.

A further difference is the requirement of the modified two-point function $\overline{\tau}$.

If we can get a uniform lower bound on the connection probability τ_s then we can get Proposition 5.8.

Proof of Lemma 5.9. We remind the reader that $\partial^{in} B_r$ is used to refer to the set $\{z \in B_r \mid d(z, B_r^c) \leq 1\}$, and that $\eta_{\langle C \rangle} = \eta \setminus \psi_*^C \eta$ and $(\eta \mid \mathcal{C}_x = C) \sim (\eta_{\langle C \rangle} \cup C)$.

For the event $U_\lambda(r, t)^c$ to hold it is required that we find two distinct clusters crossing the annulus $B_t \setminus B_r$. Each such cluster must contain a vertex in $\partial^{in} B_r$. By the Markov inequality and the Mecke equation we find that

$$\mathbb{P}[U_\lambda(r,t)^c] \leq \mathbb{E}\left[\sum_{\{x,y\}\subset\eta\cap\partial^{in}B_r}^{\neq} \mathbf{1}\{x \leftrightarrow B_t^c\}\mathbf{1}\{y \xleftrightarrow{\setminus\mathcal{C}_x} B_t^c\}\right]$$
$$= \frac{\lambda^2}{2}\int_{\partial^{in}B_r}\int_{\partial^{in}B_r} \mathbb{P}\left[x \leftrightarrow B_t^c, y \leftrightarrow B_t^c \text{ in } \xi[\eta^y \setminus \mathcal{C}_x]\right] \mathrm{d}x\mathrm{d}y.$$

We condition on the possible configurations of $\mathcal{C}_x$. Given $x, y \in \partial^{in}B_r$ we define the following measure

$$\kappa_{x,y}(dC) = \mathbf{1}\{C \cap B_t^c \neq \varnothing, y \notin C\}\mathbb{P}[\mathcal{C}(x, \xi[\eta^{x,y}]) \in \mathrm{d}C],$$

to represent all admissible components of x that reach B_t^c without connecting to y. We condition on $\mathcal{C}_x = C$:

$$\mathbb{P}[U_\lambda(r,t)^c] \leq \frac{\lambda^2}{2}\int_{\partial^{in}B_r}\int_{\partial^{in}B_r}\int \mathbb{P}[y \leftrightarrow B_t^c \text{ in } \xi[\eta^y \setminus C] \mid \mathcal{C}_x = C]\kappa_{x,y}(\mathrm{d}C)\mathrm{d}y\mathrm{d}x.$$

By the Stopping Set lemma we can write

$$\mathbb{P}\left[y \leftrightarrow B_t^c \text{ in } \xi[\eta^y \setminus C] \mid \mathcal{C}_x = C\right] = \mathbb{P}\left[y \leftrightarrow B_t^c \text{ in } \xi[\eta^y_{\langle C\rangle}]\right].$$

To use the two-arm bound we want the component $\mathcal{C}_y$ to be 'close to', but not connected to $\mathcal{C}_x = C$, which we represent by y connecting to the points that are 'deleted' in $\eta_{\langle C\rangle}$, i.e. $\psi_*^C\eta$. We will treat the deleted points more carefully later.

For some admissible C we apply the FKG inequality to the events $\{y \leftrightarrow B_t^c \text{ in } \xi[\eta^y_{\langle C\rangle}]\}$ and $\{y \leftrightarrow \psi_*^C\eta \text{ in } \xi[\eta^y \cap B_s]\}$ (which are both increasing).

$$\mathbb{P}\big[y \leftrightarrow B_t^c \text{ in } \xi[\eta^y_{\langle C\rangle}]\big] \leq \frac{\mathbb{P}\big[y \leftrightarrow B_t^c \text{ in } \xi[\eta^y_{\langle C\rangle}], y \leftrightarrow \psi_*^C\eta \text{ in } \xi[\eta^y \cap B_s]\big]}{\mathbb{P}\big[y \leftrightarrow \psi_*^C\eta \text{ in } \xi[\eta^y \cap B_s]\big]}. \tag{5.12}$$

First we bound the denominator. To reach x from y one needs to first reach a neighbor of x. It is easier to reach a neighbor of C than it is to reach a neighbor of x, since in particular $\{x\} \subset C$. So we can bound the denominator by

$$\mathbb{P}\left[y \leftrightarrow \psi_*^C\eta \text{ in } \xi[\eta^y \cap B_s]\right] \geq \mathbb{P}[y \leftrightarrow \psi_*^x\eta \text{ in } \xi[\eta^y \cap B_s]] = \inf_{x,y\in B_r} \overline{\tau}_s(x,y). \tag{5.13}$$

Notice that (5.13) is now independent of x and y.

Next we bound the numerator. We want to marginalize over C. To do so correctly, we will need to define $\eta_{\langle \mathcal{C}_x \rangle}$ and $\psi_*^{\mathcal{C}_x}\eta$ rigorously. Note that $\psi_*^{\mathcal{C}_x}\eta$ is difficult to define since by definition $\mathcal{C}_x$ should not have any additional connections.

We start with two independent instances of an independent edge marking $\xi^x = \xi^x[\eta]$ and $\overline{\xi}^x = \overline{\xi}^x[\overline{\eta}]$. Let $\overline{\mathcal{C}_x} := \mathcal{C}_x(\overline{\xi}^x)$. We now define $\widetilde{\xi}^x := \xi^x[\eta_{\langle \overline{\mathcal{C}_x} \rangle}; \overline{\mathcal{C}_x}]$, where the semicolon denotes the fact that $\overline{\mathcal{C}_x}$ already has defined edges and will (by definition) not share any edges with $\eta_{\langle \overline{\mathcal{C}_x} \rangle}$. We remark that $\widetilde{\xi}^x$ has the same law as $\xi^x \mid C_x = \overline{\mathcal{C}_x}$ (this is a direct corollary of the Stopping Set lemma). The advantage of this coupling is that we now have an explicit set of 'deleted' vertices $\psi_*^{\overline{\mathcal{C}_x}}\eta$ that we can reason about. These points can also be thought of as 'sprinkled' or 'ghost' points. This extra work is required as it is '$\psi_*^{\mathcal{C}_x}\eta$' would not be well-defined otherwise. A similar construction is used in [CD24]. We can now write:

$$\begin{aligned} &\int \mathbb{P}\left[y \leftrightarrow B_t^c \text{ in } \xi[\eta^y_{\langle C \rangle}], y \leftrightarrow \psi_*^C \eta \text{ in } \xi[\eta^y \cap B_s]\right] \kappa_{x,y}(dC) \\ &= \mathbb{P}\left[\overline{\mathcal{C}_x} \cap B_t^c \neq \varnothing, y \leftrightarrow B_t^c \text{ in } \xi[\eta^y_{\langle \overline{\mathcal{C}_x} \rangle}], y \leftrightarrow \psi_*^{\overline{\mathcal{C}_x}} \eta \text{ in } \xi[\eta^y \cap B_s]\right]. \end{aligned} \tag{5.14}$$

Note that in the process $\eta^y_{\langle \overline{\mathcal{C}_x} \rangle}$ we allow for y to be killed by $\overline{\mathcal{C}_x}$. This in particular ensures that $y \not\leftrightarrow \overline{\mathcal{C}_x}$, as is required by our earlier choice of $\kappa_{x,y}$.

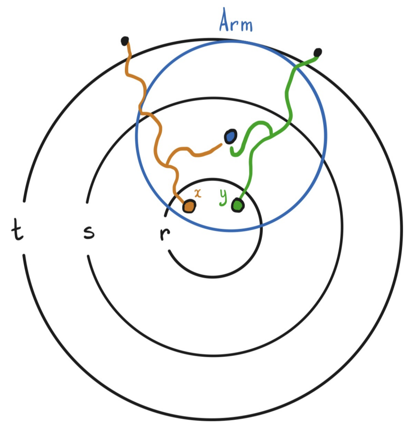

Figure 5-3: Arm event centered around a point $z \in \psi_*^{\overline{\mathcal{C}_x}}\eta$.

If the event in (5.14) holds we know that at least one 'deleted' point $z \in \psi_*^{\overline{\mathcal{C}_x}}\eta$ connects to B_t^c via two disconnected paths: once through the x component, and once through the y component.

In particular, the event $Arm_z(t-s)$ should hold for at least one $z \in \psi_*^{\overline{\mathcal{C}_x}}\eta$. To apply the Arm upper bound from Proposition 5.6 we will first have to apply the Mecke equation (in the reverse direction) to avoid problems arising from x or y being pivotal themselves. Hence, we get the following bound:

$$
\begin{aligned}
\frac{\lambda^2}{2}\int_{\partial^{in}B_r}\int_{\partial^{in}B_r}&\mathbb{P}\left[\overline{C_x}\cap B_t^c\neq\varnothing, y\leftrightarrow B_t^c \text{ in } \xi[\eta^y_{\langle\overline{C_x}\rangle}], y\leftrightarrow\psi_*^{\overline{C}_x}\eta \text{ in } \xi[\eta^y\cap B_s]\right]\mathrm{d}y\mathrm{d}x\\
&=\mathbb{E}\left[\sum^{\neq}_{\{x,y\}\subset\eta\cap\partial^{in}B_r} 1\left\{\overline{C}_x\cap B_t^c\neq\varnothing, y\leftrightarrow B_t^c \text{ in } \eta_{\langle\overline{C}_x\rangle}, y\leftrightarrow\psi_*^{\overline{C}_x}\eta,\right\}\right]\\
&\leq\mathbb{E}\left[\sum^{\neq}_{\{x,y\}\subset\eta\cap\partial^{in}B_r}\sum_{z\in\psi_*^{\overline{C}_x}\eta\cap B_s} 1\left\{Arm_z(B_{t-s}(z))\right\}\right]
\end{aligned}
$$

Let us write $X := |\eta\cap\partial^{in}B_r|$. The summand in the above display does not depend on x or y. Hence, we can replace the sum by the number of (unordered) Poisson pairs in $\eta\cap\partial^{in}B_r$ which is equal to $\frac{1}{2}(X^2-X)$. Next, the point process $\psi_*^{\overline{C}_x}\eta\cap B_s$ can be dominated by a Poisson point process of density $\lambda\mathbf{1}_{B_s}$. We then use the Mecke equation to find:

$$
\begin{aligned}
&\mathbb{E}\left[\sum^{\neq}_{x,y\in\eta\cap\partial^{in}B_r}\sum_{z\in\psi_*^{\overline{C}_x}\eta\cap B_s}\mathbf{1}\left\{Arm_z(B_{t-s}(z))\right\}\right]\\
&=\mathbb{E}\left[\sum_{z\in\psi_*^{\overline{C}_x}\eta\cap B_s}\frac{1}{2}\left(X^2-X\right)\mathbf{1}\left\{Arm_z(B_{t-s}(z))\right\}\right]\\
&\leq\lambda\int_{B_s}\mathbb{E}\left[X^2\mathbf{1}\{Arm_z(B_{t-s}(s))\}\right]\mathrm{d}s.
\end{aligned}
$$

In the above inequality we use the fact that $|\eta\cap\partial^{in}B_r|$ increases by 1 whenever $z\in\partial^{in}B_r$. We simply add 1 to X no matter the location of z. Furthermore, we bound $\frac{1}{2}(X^2+X)\leq X^2$. Next, we apply the Hölder inequality with parameters $\alpha\in[0,\infty]$ and $\frac{\alpha}{\alpha-1}$.[4] Then, using Lemma 5.4 to bound the Poisson moment in the second inequality:

$$
\begin{aligned}
\lambda\int_{B_s}\mathbb{E}\left[X^2\mathbf{1}\{Arm_z(B_{t-s}(s))\}\right]\mathrm{d}s&\leq\lambda\int_{B_s}\mathbb{E}[X^{2\alpha}]^{\frac{1}{\alpha}}\mathbb{P}[Arm_z(B_{t-s}(z))]^{\frac{\alpha-1}{\alpha}}\mathrm{d}s\\
&\leq\lambda^3|B_s||\partial^{in}B_r|^2\exp\left(\frac{2\alpha}{\lambda|\partial^{in}B_r|}\right)\mathbb{P}[Arm(t/2)]^{\frac{\alpha-1}{\alpha}},
\end{aligned}
\tag{5.15}
$$

where we use the fact that $\mathbb{P}[Arm_z(B_{t-s}(z))]\leq\mathbb{P}[Arm(t/2)]$. Remember

Recall that (5.15) is an upper bound on the (integral of the) numerator of (5.12).

[4]When applying Hölder's inequality we need to be careful to include the mark of z in $\mathbb{E}$.

Together with the lower bound on the denominator (5.13) we find that

$$\mathbb{P}[U_\lambda(r,t)^c] \leq c_4 \exp\left(\frac{2\alpha}{\lambda|\partial^{in} B_r|}\right) \frac{\lambda^3 r^{2d-2} s^d}{\inf_{x,y\in B_r} \overline{\tau}_s(x,y)} \mathbb{P}[Arm(t/2)]^{\frac{\alpha-1}{\alpha}}.$$

Since $|B_s| \asymp s^d$ and $|\partial^{\text{in}} B_r| \asymp r^{d-1}$, this gives us the desired bound. ■

We do not yet have a good lower bound on $\inf_{x,y\in B_r} \overline{\tau}_{2r}(x,y)$. We will construct a lower bound via an iterative procedure based on the following lemma. We introduce the following notation for some $z \in \mathbb{R}^d$: let $U_\lambda(r,t;z)$ represent the uniqueness event $U_\lambda(r,t)$ centered at z.

We remind the reader that (**A2**) refers to the existence of some $c, \delta > 0$ such that $\mathbb{P}[B_s \leftrightarrow \infty] \geq 1 - cs^{-\delta}$ holds for all $s \geq 1$. The following lemma (along with the proof) is adapted from [CMT24].

Lemma 5.11. *Let $\lambda > \lambda_c$ and $\delta > 0$ as in (**A2**). Let $m, u \in \mathbb{R}_{\geq 1}$ such that $m \geq u^{1+\delta}$. Then*

$$\mathbb{P}[U_\lambda(u,m)^c] \leq \frac{\delta}{u^\delta} \qquad \Longrightarrow \qquad \forall a, b \in B_{u^{1+\delta}} : \overline{\tau}_{2m}(a,b) \geq \delta.$$

Remark 5.12. Note that Lemma 5.11 does not hold if we can choose the mark of a and b in general, as it might be possible to choose arbitrarily unfavorable marks.

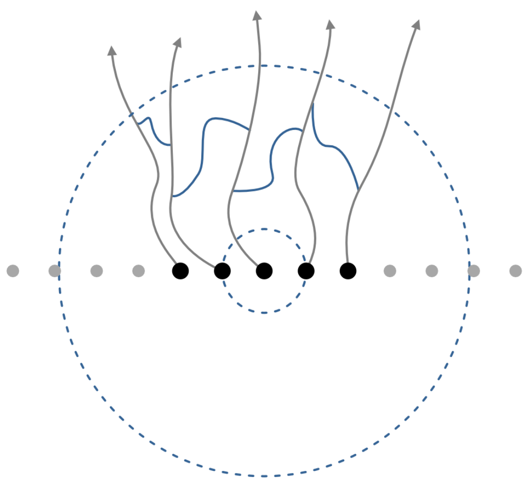

Figure 5-4: Gluing via paths to infinity and uniqueness events. The black dots represent the balls $B_{u/2}(x_i)$. The gray arrows represent paths to infinity. The blue paths in between the gray paths exist due to uniqueness. For sake of clarity I did not include all paths.

Proof. Let $c, \delta > 0$ be as in (**A2**). We can assume furthers, without loss of generality, that δ is sufficiently small such that $\theta(\lambda)^2 4^{-3c2^\delta} \geq 4\delta$ holds.

Now let $a, b \in B_{u^{1+\delta}}$. Let $n = \lceil 4u^\delta \rceil \leq 5u^\delta$. We write $x_i := a + \frac{i}{n}(b-a)$ for all $i \in [\![0, n]\!]$. Then $x_0 = a$ and $x_n = b$. Then $\|x_i - x_{i+1}\|_2 \leq \frac{u}{2}$.

We assume that $a \leftrightarrow \infty$ and $b \leftrightarrow \infty$ and $B_{u/2}(x_i) \leftrightarrow \infty$ for all $i \in [\![1, n-1]\!]$. Then either $a \leftrightarrow b$ in B_{2m} (as demonstrated in Figure 5-4), or, for some $i \in [\![1, n-1]\!]$ we find that $U(u, m; x_i)^c$ occurs. Thus, by the union bound:

$$\mathbb{P}[a \leftrightarrow \infty, b \leftrightarrow \infty, \forall i : B_{u/2}(x_i) \leftrightarrow \infty] \leq \overline{\tau}_{2m}(a, b) + \sum_{i \in [\![1, n-1]\!]} \mathbb{P}[U_\lambda(u, m; x_i)^c].$$

Note that for any $q \in [0, \frac{1}{2}]$ and $p > 0$ it holds that $(1-q)^p \geq 4^{-qp}$. By FKG-inequality, Assumption (**A2**), and the assumption that $P[U_\lambda(u, m)^c] \leq \frac{\delta}{u^\delta}$:

$$\begin{aligned}
\overline{\tau}_{2m}(a, b) &\geq \mathbb{P}[a \leftrightarrow \infty]\mathbb{P}[b \leftrightarrow \infty]\mathbb{P}[B_{u/2} \leftrightarrow \infty]^{n-1} - (n-1)\mathbb{P}[U_\lambda(u, m)^c] \\
&\geq \theta(\lambda)^2 (1 - c2^\delta u^{-\delta})^{3u^\delta} - 3u^\delta \mathbb{P}[U_\lambda(u, m)^c] \\
&\geq \theta(\lambda)^2 4^{-3c2^\delta} - 3\delta.
\end{aligned}$$

Thus, by our assumption on δ that $\theta(\lambda)^2 4^{-3c2^\delta} \geq 4\delta$,

$$\overline{\tau}_{2m}(a, b) \geq \delta.$$

■

The approach in the following proof is to use Lemma 5.9 together with Lemma 5.11. Lemma 5.9 gives an upper bound on $U(r, t)^c$, which we may use in order to lower bound $\overline{\tau}_\lambda(x, y)$ with Lemma 5.11. Each time we increase the scale, allowing us to iterate this procedure and get a bound on any (sufficiently large) scale.

Proof of Proposition 5.8. Let μ_d be the volume of the unit ball in d dimensions. Let $\delta \in (0, \frac{1}{4})$ so that it is compatible with Lemma 5.11. Let us restate Lemma 5.9, replacing $\mathbb{P}[Arm(t/2)]$ with the bound from Corollary 5.7:

$$\mathbb{P}[U_\lambda(r, t)^c] \leq C \exp\left(\frac{2\alpha}{\lambda \mu_d r^{d-1}}\right) \frac{r^{2d-2} s^d}{\inf_{x, y \in B_r} \overline{\tau}_s(x, y)} \log(t)^{\frac{\alpha-1}{2\alpha}} t^{-\frac{\alpha-1}{2\alpha}}. \tag{5.16}$$

Now choose $s = 2r$. We write $\varepsilon(r, t) := (\lambda \mu_d \log(t) r^{d-1})^{-\frac{1}{2}}$. Then we choose $\alpha =$

$\frac{1}{2}\varepsilon(r,t)^{-1}$ and drop the $-\varepsilon(r,t)$ exponent on the $\log(t)$ term to find:

$$\mathbb{P}[U_\lambda(r,t)^c] \leq C \exp\left(\sqrt{\frac{\log(t)}{\lambda \mu_d r^{d-1}}}\right) \frac{2^d r^{3d-2}}{\inf_{x,y \in B_r} \overline{\tau}_{2r}(x,y)} \log(t)^{\frac{1}{2}} t^{-\frac{1}{2}+\varepsilon(r,t)}. \tag{5.17}$$

As we are using the above equation repeatedly we let me remind the reader that we assume that $t \geq 2r$ and $t > e$.

We will now define r_0 and t_0 to ensure our induction step later works. First, we fix $r_0 \geq 2$ such that $\varepsilon(r_0, r_0^{1+\delta}) \leq \frac{1}{16}$. Next, we can find $t_0 \geq \max(r_0^{1+\delta}, 2(C/\delta^2)^{\frac{1}{d}})$ such that $\mathbb{P}[U_\lambda(r_0,t_0)^c] \leq \frac{\delta}{r_0^\delta}$ by (5.17). We can now use Lemma 5.11 to find that $\inf_{a,b \in B_{r_0^{1+\delta}}} \overline{\tau}_{2t_0}(a,b) \geq \delta$.

We further require that for all $t \geq t_0$ it holds that

$$t^{\frac{1}{8}} \geq \log(t)^{\frac{1}{2}} \exp\left(\sqrt{\frac{\log(t)}{\lambda \mu_d r_0^{d-1}}}\right), \tag{5.18}$$

which is possible by inspection. This bound also holds for larger values of r_0. In particular, it will hold for subsequent induction steps.

We now choose $r_1 := r_0^{1+\delta}$. We use (5.16) with $r = r_1$, $s = 2t_0$ and $t = t_1 := t_0^{64d}$ together with $\inf_{a,b \in B_{r_0^{1+\delta}}} \overline{\tau}_{2t_0}(a,b) \geq \delta$. It follows that

$$\begin{aligned}\mathbb{P}[U_\lambda(r_1,t_1)^c] &\leq \left[t_1^{-\frac{1}{8}} \exp\left(\sqrt{\frac{\log(t_1)}{\lambda \mu_d r_1^{d-1}}}\right) \log(t_1)^{\frac{1}{2}}\right] \left[t_1^{-\frac{1}{16}+\varepsilon(r_1,t_1)}\right] \left[t_1^{-\frac{1}{16}} 2^d \frac{C}{\delta} r_1^{2d-2} t_0^d\right] t_1^{-\frac{1}{4}} \\ &\leq \frac{\delta}{t_1^{1/4}}.\end{aligned}$$

To clarify, the $t_1^{-\frac{1}{2}}$ term is split into four parts, each separated by a square bracket. We inspect each individually going from left to right.

1. By (5.18) we ensured that $t_1^{-\frac{1}{8}}$ dominates the log terms, and so the first bracket is less than one.

2. We reserve $t_1^{-1/16}$ to cancel the $t_1^{\varepsilon(r_1,t_1)}$ term, thus the second bracket is less than one.

3. We use $t_1^{1/16} = t_0^{4d}$, which must be larger than $2^d(C/\delta^2)r_1^{(2d-2)}t_0^d$ by definition. This term gives use the extra δ.

Finally, this leaves us with $\frac{\delta}{t_1^{1/4}}$.

Now we repeat this argument to bootstrap this bound. Since $\frac{\delta}{t_1^{1/4}} \leq \frac{\delta}{t_1^{\delta}}$, we can use Lemma 5.11 again to find that $\inf_{a,b \in B_{r_i^{1+\delta}}} \overline{\tau}_{2t_i}(a,b) \geq \delta$. For general, for $i \in \mathbb{N}$ we define $r_i := r_{i-1}^{1+\delta}$ and $t_i := t_{i-1}^{64d}$. And so, by the same reasoning as above, we find:

$$\mathbb{P}[U_\lambda(r_i, t_i)^c] \leq \frac{\delta}{t_i^{1/4}}.$$

We can choose $\chi \leq \frac{\log(1+\delta)}{\log(2d+\delta)}$ so that $r_i \leq \exp(\log(t_i)^\chi) = h(t_i)$. ■

Corollary 5.13. *Let $\lambda > \lambda_c$. Let $\delta_0 > 0$ be as in* (**A2**)*. Then for all R sufficiently large and all $x, y \in B_R$ it holds that*

$$\tau_{2R}(x,y) \geq \delta.$$

Proof. We write Proposition 5.8 as $\mathbb{P}[U_\lambda(r, H(r))^c] \leq c_2 H(r)^{-1/4}$. We first check the conditions of Lemma 5.11. Let $\delta > 0$. We require that $H(r) \geq r^{1+\delta}$. Since H grows superpolynomially this condition holds for sufficiently large r.

Next we require that $\mathbb{P}[U_\lambda(r, H(r))] \leq c_2 H(r)^{-1/4} \leq \frac{\delta}{r^\delta}$. By rearranging this equation we can see that again it must hold for sufficiently large r due to the superpolynomial growth of H.

Now we may apply Lemma 5.11, and the statement holds. ■

Chapter 6

Grimmett-Marstrand

A very important result in percolation theory is the that in the supercritical phase percolation should still occur in sufficiently large subsets of space. This was first characterized by Grimmett and Marstrand in [GM90]. They show that in the supercritical phase we still percolate in sufficiently big slabs of the form $\mathrm{Slab}_l := \mathbb{R}^2 \times [-l, l]^{d-2}$.

In this chapter we will prove two versions of this fact, first a qualitative version which follows quickly from the results of the previous chapters. Next, a quantitative version, which provides more insight into how large the slab needs to be.

For the rest of this chapter we assume (**A2**) and that $d \geq 3$.

6.1 Statement

We will take a slightly different approach here more focused on developing techniques related to sprinkling. This will have the advantage of giving us a quantitative result, rather than just an existence result.

We take the ideas in the following section from [DKT21], in particular we use the 'seedless' renormalization scheme. We will use c to refer to any strictly positive constant to simplify notation. To make this argument work we need to additionally assume that ψ is spherically symmetric.

Theorem 6.1 (Quantitative Grimmett-Marstrand)**.** *Fix $d \geq 3$ and $\lambda > 0$. Assume that ψ spherically symmetric. There exists a constant $C = C(d) > 0$ such that the following holds. Assume for some $\varepsilon > 0$ and $1 \leq k < K < n < N < \infty$ such that $K \leq \varepsilon^2 n$ the following assumptions hold:*

(a) $\int_{\mathbb{M}} \mathbb{P}[o_m \leftrightarrow \Lambda_n^c]\rho(\mathrm{d}m) \geq \varepsilon$

(b) $\mathbb{P}[\Lambda_k \leftrightarrow \Lambda_N^c] \geq 1 - \exp(-1/\varepsilon)$

(c) $\mathbb{P}[U_\lambda(k,K)^c] \leq \exp(-1/\varepsilon)$ *and* $\mathbb{P}[U_\lambda(n,N)^c] \leq \exp(-1/\varepsilon)$.

Then, we have

$$\mathbb{P}_{\lambda+C\varepsilon}[o \xleftrightarrow{\mathrm{Slab}_{2N}} \infty] \geq \varepsilon/2.$$

Note that for $\lambda > \lambda_c$ we can fulfill these assumptions. We get (a) immediately by, i.e. by choosing $\varepsilon \in (0, \theta(\lambda)/2)$. To ensure (b) holds we choose k such that

$$\mathbb{P}[\Lambda_k \leftrightarrow \Lambda_N^c] \overset{(\mathbf{A2})}{\geq} 1 - ck^{-\delta} \geq 1 - \exp(-1/\varepsilon)$$

holds. We can solve for k by taking the $(-\delta)$-th root of $ck^{-\delta} \leq \exp(-1/\varepsilon)$ which yields $k \geq c^{1/\delta}\exp(\frac{1}{\delta\varepsilon})$.

Recall that we defined $H(r) = \exp(\log(r)^{1/\chi})$ for $\chi \in (0,1)$.

We now set $(k,K,n,N) := (k, H(k), H^2(k), H^3(k))$. Thus, by Proposition 5.8, we have

$$\mathbb{P}[U_\lambda(k,K)^c] \leq c_2 K^{-1/4} \leq c_2 \exp\left(\log(k)^{1/\chi}\right)^{-1/4} \leq \exp(-\frac{1}{\varepsilon}),$$

which can be rearranged to yield $k \geq \exp(4^\chi(1/\varepsilon - \log(c_2))^\chi)$.

Finally, choosing

$$k = \max\left\{c^{1/\delta}\exp\left(\frac{1}{\delta\varepsilon}\right), \exp\left(4^\chi(1/\varepsilon - \log(c_2))^\chi\right)\right\}$$

ensures items (b) and (c) hold.

Lemma 6.2 (Square root trick)**.** *Let $n \in \mathbb{Z}_{\geq 1}$. Let $A_1, \dots, A_n$ be increasing events. Then,*

$$\max_{i \in [\![1,n]\!]} \mathbb{P}[A_i] \geq 1 - \left(1 - \mathbb{P}\left[\bigcup_{i \in [\![1,n]\!]} A_i\right]\right)^{1/n}.$$

Proof. By the FKG inequality, and the fact that A_i^c are decreasing, we know that

$$\mathbb{P}\left[\bigcup_{i \in [\![1,n]\!]} A_i\right] \leq 1 - \prod_{i \in [\![1,n]\!]} \mathbb{P}[A_i^c] \leq 1 - \left(1 - \max_{i \in [\![1,n]\!]} \mathbb{P}[A_i]\right)^n.$$

The result follows by solving for $\max_{i\in[\![1,n]\!]}\mathbb{P}[A_i]$. ■

6.1.1 Points

We remind the reader of the notation $\Lambda_n := [-n, n]^d$. We will require two lemmas before we proceed with the proof of Theorem 6.1.

1. Lemma 6.3 shows that (with the same conditions as Theorem 6.1) a sufficiently large set of inserted vertices will, with high probability, connect to one of the quarter-faces of Λ_n (defined below).

2. Lemma 6.5 says that if a connection event with inserted vertices holds with sufficiently high probability, then an associated connection event involving sprinkling will also hold with high probability.

In the upcoming proofs we will use the following notation for the sake of compactness:

$$\{A \overset{K}{\longleftrightarrow} B\} := \{A \leftrightarrow B \text{ in } \xi \cap K\},$$

where $K \subseteq \mathbb{R}^d$ and A and B are either point sets, point measures or subsets of $\mathbb{R}^d$.

We will combine these lemmas to perform the 'seedless' renormalization described in [DKT21]. Let $i \in [\![1, d]\!]$. We define the i-th 'quarter-face' of Λ_N as:

$$F_i(N) := \{(x_1, \ldots, x_d) \mid x_i \in [N, N+1), \forall j \neq i : x_j \in [0, N+1)\}.$$

Note that the name 'quarter-face' is only accurate in $3d$. In d dimensions each face has 2^{d-1} 'quarters', meaning that the hypercube has $2^d d$ quarter-faces in total.

Lemma 6.3. *Assume* (a), (b) *and* (c) *hold. Then there exists some* $c, \beta > 0$ *(depending only on* ψ *and* d*) such that for a.e.-* $C \sim \mathcal{C}(\eta^o, o)$ *with* $\operatorname{diam}(C) \geq n$ *we have for all* i

$$\mathbb{P}[C \overset{\Lambda_{N+1}}{\longleftrightarrow} F_i(N) \text{ in } \xi[\eta \cup C]] \geq 1 - \beta \exp(-c/\varepsilon).$$

Note, here we 'overlay' C *and* η*.*

Proof. Throughout this proof we reserve the symbols $\tilde{c}_1, \tilde{c}_2, \ldots$ to refer to strictly positive constants that depend only on d and ψ.

By the Square root trick and assumption (b) we find that for all $i \in [\![1, d]\!]$:

$$\mathbb{P}[\Lambda_k \overset{\Lambda_N}{\longleftrightarrow} F_i(N)] \geq 1 - \exp(-1/(2^d d\varepsilon)). \tag{6.1}$$

This is the only point where we require the assumption that ψ is spherically symmetric.

We assume without loss of generality that $C \subset \Lambda_n$. This can be done as $C \cap \Lambda_n$ connecting to $F_i(N)$ implies that all of C will still make the same connection. Consider $x_1, \dots, x_l \in C$ such that $Q_j := x_j + \Lambda_{K+1}$ are all disjoint and contained in Λ_{n+1}. We also define a smaller box $Q'_j := x_j + \Lambda_k$. We can and do write $l = \lceil \tilde{c}_1/\varepsilon^2 \rceil$ for some constant $\tilde{c}_1$. We define the following events

$$\begin{aligned} E_j &:= \{x_j \leftrightarrow Q_j^c\} \cap U_\lambda(k, K; x_j) \\ B_j &:= \{Q'_j \not\leftrightarrow \Lambda_N^c\}. \end{aligned}$$

By (a) and (c) together we have

$$\begin{aligned} \mathbb{P}[E_j] &\geq \mathbb{P}[x_j \leftrightarrow Q_j^c] - \mathbb{P}[U_\lambda(k, K)^c] \\ &\geq \varepsilon - \exp(-1/\varepsilon) \geq \varepsilon/2. \end{aligned}$$

Since each the events E_j lives on $\Lambda_{K+1} + x_j$, they are independent, thus

$$\mathbb{P}\left[\bigcup_{j \in [\![1,l]\!]} E_j\right] \geq 1 - (1 - \varepsilon/2)^l \geq 1 - 2e^{-\tilde{c}_2 \varepsilon l} \geq 1 - 2e^{\tilde{c}_3/\varepsilon},$$

where we use $1 - x \leq e^{-x}$.

Next we bound $\mathbb{P}[B_j]$, by observing that at least half of the faces of $x_j + \Lambda_N$ are outside of Λ_N. It follows from (6.1) that

$$\mathbb{P}[B_j] \leq \mathbb{P}\left[Q'_j \not\leftrightarrow (x_j + \partial^{\text{ext}} \Lambda_N) \cap \Lambda_N^c\right] \overset{(6.1)}{\leq} \exp\left(-\frac{1}{2^d d \varepsilon}\right).$$

By the union bound we find

$$\mathbb{P}\left[\bigcup_{j \in [\![1,l]\!]} B_j\right] \leq l \exp\left(-\frac{1}{2^d d \varepsilon}\right) \leq \exp(-\tilde{c}_4/\varepsilon),$$

where the last inequality uses the fact that $l \leq \tilde{c}_1/\varepsilon^2 + 1$. Now assume there exists at least one i such that $E_j \setminus B_j$ occurs. Then we know that $x_j \leftrightarrow Q_j^c$ and $Q'_j \leftrightarrow \Lambda_N^c$ (from B_j^c) both happen. Furthermore, we know that these two paths must be connected. In particular $C \leftrightarrow \Lambda_N^c$ occurs. And so:

$$\mathbb{P}[C \leftrightarrow \Lambda_N^c] \geq \mathbb{P}\left[\bigcup_{j\in[\![1,l]\!]} E_j \setminus B_j\right] \geq \mathbb{P}\left[\bigcup_{j\in[\![1,l]\!]} E_j\right] - \mathbb{P}\left[\bigcup_{j\in[\![1,l]\!]} B_j\right] \geq 1 - 2e^{-\tilde{c}_5/\varepsilon}.$$

Finally, by the above display, (6.1) and (c)

$$\begin{aligned}\mathbb{P}[C \overset{\Lambda_N}{\longleftrightarrow} F_i(N)] &\geq \mathbb{P}[\Lambda_n \leftrightarrow F_i(N), C \leftrightarrow \Lambda_N^c, U_\lambda(n,N)] \\ &\geq 1 - \beta e^{-\tilde{c}_6/\varepsilon},\end{aligned}$$

for some $\beta > 0$. ■

Remark 6.4. By inspection of the above proof, in particular that we assume $C \subset \Lambda_n$, we can also see that we need to use at least one neighbor of C, thus

$$\mathbb{P}[\psi_*^C \eta \overset{\Lambda_{N+1}}{\longleftrightarrow} F_i(N) \text{ in } \xi[\eta]] \geq 1 - \beta \exp(-c/\varepsilon).$$

This will allow us to use the result of Lemma 6.3 for Lemma 6.5 below.

For Lemma 6.3 to be useful we require a way to convert probabilities between 'overlaid' points and probabilities on ξ_λ. We want to be able to say that if some connection event is very likely for a given point-set A that it continues to be likely if these points can only connect to η via some extra sprinkling $\widetilde{\eta}$. This can be interpreted as 'closing all the edges of A'.

We will use the Stopping Set lemma. We remind the reader of the notation $\xi[\eta,\eta']$ defined in (2.1). It consists of the edges between η and η' and η and itself, but not η' and itself. The following we adapt from [DKT21, Lemma 10].

Lemma 6.5. *For any $\gamma > 0$, $\delta \in (0, \frac{1}{4})$ and $\lambda > 0$ there exists some $\kappa > 0$ (depending on γ, δ and λ, but not ε) such that for any $\varepsilon \in (0,\delta)$ and any thinning function g with compact support and any finite set $A \subset \mathbb{X}$ satisfying the following relation:*

$$\mathbb{P}[\psi_*^A \eta \leftrightarrow g_* \eta \text{ in } \xi[\eta]] \geq 1 - \exp(-\gamma/\varepsilon), \tag{6.2}$$

we have

$$\mathbb{P}\big[\psi_*^A \widetilde{\eta} \leftrightarrow g_* \eta \text{ in } \xi[\eta_{\langle A\rangle}, \psi_*^A \widetilde{\eta}]\big] \geq 1 - \delta,$$

where $\eta \sim \lambda{-}ppp$ and $\widetilde{\eta} \sim \kappa\varepsilon$-ppp.

Proof. Let $\varepsilon > 0$ so that (6.2) holds. Note that

$$\mathbb{P}[\psi_*^A \eta \leftrightarrow g_* \eta \text{ in } \xi[\eta]] = \mathbb{P}[\psi_*^A \eta \leftrightarrow g_* \eta \text{ in } \xi[\eta_{\langle A \rangle}, \psi_*^A \eta]]$$

In words this means that it suffices to consider paths that start at a neighbor of A and never return to any neighbor of A.

Furthermore, we can now see that we can sample $\eta_{\langle A \rangle}$ separately from $\psi_*^A \eta$. To make this explicit we introduce a new Poisson point process $\tilde{\eta}_L$ of (varying) intensity $L > 0$, independent of everything else. We can then write

$$\mathbb{P}\left[\psi_*^A \eta \leftrightarrow g_* \eta \text{ in } \xi[\eta_{\langle A \rangle}, \psi_*^A \eta]\right] = \mathbb{P}\left[\psi_*^A \tilde{\eta}_\lambda \leftrightarrow g_* \eta \text{ in } \xi[\eta_{\langle A \rangle}, \psi_*^A \tilde{\eta}_\lambda]\right].$$

We define the following set of points that share an edge with A and are connected to $g_* \eta$ without using any other points that are connected to A:

$$W_L := \left| \left\{ x \in \tilde{\eta}_L \;\middle|\; x \sim A, x \leftrightarrow g_* \eta \text{ in } \xi[\eta_{\langle A \rangle} \cup \{x\}] \right\} \right|.$$

Let $L > \lambda$ and notice that if we take a λ/L thinning of $\psi_*^A \tilde{\eta}_L$ that we recover $\psi_*^A \tilde{\eta}_\lambda$. In particular, one way the connection event $\psi_*^A \tilde{\eta}_\lambda \leftrightarrow g_* \eta$ can fail is that $W_L < n$ for some $n \geq 1$ and each of vertices die in the λ/L-thinning. In symbols:

$$\mathbb{P}[\psi_*^A \tilde{\eta}_\lambda \not\leftrightarrow g_* \eta] \geq \mathbb{P}[W_L < n] \left(1 - \frac{\lambda}{L}\right)^{n-1}. \tag{6.3}$$

On the other hand, one way the connection can take place for $\psi_*^A \tilde{\eta}_{\kappa\varepsilon}$ is that $W_L \geq n$ and at least one vertex survives a $\frac{\kappa\varepsilon}{L}$-thinning. In symbols, we get:

$$\begin{aligned}
\mathbb{P}\left[\psi_*^A \tilde{\eta}_{\kappa\varepsilon} \leftrightarrow g_* \eta \text{ in } \xi[\eta_{\langle A \rangle}, \psi_*^A \tilde{\eta}_{\kappa\varepsilon}]\right] &\geq \mathbb{P}[W_L \geq n] \left(1 - \left(1 - \frac{\kappa\varepsilon}{L}\right)^n\right) \\
\text{(Using (6.3))} \qquad &\geq \left(1 - \frac{\mathbb{P}[\psi_*^A \tilde{\eta}_\lambda \not\leftrightarrow g_* \eta]}{(1 - \frac{\lambda}{L})^{n-1}}\right) \left(1 - \left(1 - \frac{\kappa\varepsilon}{L}\right)^n\right) \qquad (6.4) \\
\text{(Using (6.2))} \qquad &\geq \left(1 - \frac{\exp(-\gamma/\varepsilon)}{(1 - \frac{\lambda}{L})^{n-1}}\right) \left(1 - \left(1 - \frac{\kappa\varepsilon}{L}\right)^n\right).
\end{aligned}$$

The proof now follows by optimizing over n. We will see that $n = L\frac{\frac{\gamma}{\varepsilon} + \log\left(\frac{\kappa\varepsilon}{\lambda}\right)}{\kappa\varepsilon + \lambda}$ suffices (but is not strictly optimal). This choice of n is found by taking the derivate of $1 - \frac{\exp(-\gamma/\varepsilon)}{(1-\frac{\lambda}{L})^{n-1}} - (1 - \frac{\kappa\varepsilon}{L})^n$ and setting the result equal to 0, i.e. we ignore the mixed term from (6.4). By substituting n in (6.4) and letting $L \to \infty$ we find that

$$\begin{aligned}
&\mathbb{P}\left[\psi_*^A \tilde{\eta}_{\kappa\varepsilon} \leftrightarrow g_*\eta \text{ in } \xi[\eta_{\langle A\rangle}, \psi_*^A \tilde{\eta}_{\kappa\varepsilon}]\right] \\
&\quad \geq \left(1 - \exp\left(\frac{-\kappa\gamma + \lambda\log(\frac{\kappa\varepsilon}{\lambda})}{\kappa\varepsilon+\lambda}\right)\right)\left(1 - \exp\left(\frac{-\kappa\gamma - \kappa\varepsilon\log(\frac{\kappa\varepsilon}{\lambda})}{\kappa\varepsilon+\lambda}\right)\right) \\
&\quad \geq 1 - \exp\left(-\frac{\gamma\kappa}{\kappa\varepsilon+\lambda}\right)\left(\exp\left(\frac{\lambda\log(\frac{\kappa\varepsilon}{\lambda})}{\kappa\varepsilon+\lambda}\right) + \exp\left(\frac{-\kappa\varepsilon\log(\frac{\kappa\varepsilon}{\lambda})}{\kappa\varepsilon+\lambda}\right)\right).
\end{aligned}$$

We can bound the sum of exponentials by rewriting $x = \frac{\kappa\epsilon}{\lambda}$, which gives $x^{\frac{1}{x+1}} + x^{-\frac{x}{x+1}}$. Using the weighted inequality of arithmetic and geometric means, i.e. $ta + (1-t)b \geq a^t b^{1-t}$ with $a = x$, $b = 1$ and $t = \frac{1}{x+1}$ gives $2\frac{x}{1+x} \geq x^{\frac{1}{x+1}}$, which can be rearranged to yield

$$x^{\frac{1}{x+1}} + x^{-\frac{x}{x+1}} \leq 2$$

for all $x > 0$. This inequality is sharp at $x = 1$.

Finally, by the above display, and using that $\varepsilon \leq \delta$ we find that:

$$\mathbb{P}\left[\psi_*^A \tilde{\eta}_{\kappa\varepsilon} \leftrightarrow g_*\eta \text{ in } \xi[\eta_{\langle A\rangle}, \psi_*^A \tilde{\eta}_{\kappa\varepsilon}]\right] \geq 1 - 2\exp\left(-\frac{\gamma\kappa}{\kappa\delta+\lambda}\right).$$

Choosing κ as follows

$$\frac{-\lambda\log(\delta/2)}{\gamma + \delta\log(\delta/2)} \leq \kappa$$

is sufficient. ■

6.1.2 Proof of Theorem 6.1

Proof of Theorem 6.1. The overall structure of this proof will be to build an iterative exploration of Slab_{3N} by tiling it with overlapping boxes that we index with $\mathbb{Z}^2$. For every $x \in \mathbb{Z}^2$ we write

$$\Lambda_x := Nx + \Lambda_N \qquad \text{and} \qquad \widetilde{\Lambda}_x := Nx + \Lambda_{3N}.$$

Let η be a λ-Poisson point process in Slab_{2N}. For every $x \in \mathbb{Z}^2$ we let η_x be a $\kappa\varepsilon$-Poisson point process in $\widetilde{\Lambda}_x$, where κ is chosen later. The η_x are all independent of each other and from η. We will prove that the event $o \leftrightarrow \infty$ in $\eta_{\text{total}} := \eta \cup (\bigcup_x \eta_x) \cup \{o\}$ occurs with probability at least $\varepsilon/2$.

Fix any ordering of the edges $E(\mathbb{Z}^2)$. Let $\eta_0 := \eta \cup \{o\}$. Let $A_0 := \{o\}$ and $B_0 := \varnothing$. Let $t \geq 0$ and $X_t := (A_t, B_t)$. To define X_{t+1} given X_t we consider the next edge

$e \in E(\mathbb{Z}^2)$, such that one endpoint is in A_t, and the other endpoint is in $(A_t \cup B_t)^c$. As such, we view A_t as the open cluster and B_t as the boundary.

Let e be the next admissible edge and let x be the endpoint of e not in $A_t \cup B_t$, then

$$\eta_{t+1} := \eta_t \cup \eta_x \qquad \text{and} \qquad X_{t+1} := \begin{cases} (A_t \cup \{x\}, B_t) & \text{if } o \leftrightarrow \Lambda_x \text{ in } \eta_{t+1} \\ (A_t, B_t \cup \{x\}) & \text{else} \end{cases}.$$

We have the following properties two properties:

- $\eta_\infty := \bigcup_{t \geq 0} \eta_t \subset \eta_{\text{total}} := \eta_0 \cup \bigcup_{x \in \mathbb{Z}^2} \eta_x$.
- If X_t percolates then $o \overset{\eta_\infty}{\longleftrightarrow} \infty$.

We now wish to prove $\mathbb{P}[X \text{ percolates} \mid o \leftrightarrow \Lambda_o^c \text{ in } \eta \cup \{o\}] \geq 1/2$.

We require the following classic result from [Gri99, Lemma 7.24].

Lemma 6.6. *If for the random exploration $X_t = (A_t, B_t)$ there exists some $q > p_c^{\text{site}}(\mathbb{Z}^2)$ such that*

$$\mathbb{P}[B_{t+1} = B_t \mid X_0, \dots, X_t] \geq q \qquad \text{a.s. for all } t \geq 0,$$

then $\mathbb{P}[|A_\infty| = \infty] \geq c(q) > 0$ where $c(q) \xrightarrow{q \nearrow 1} 1$. If $|A_\infty| = \infty$ we say that the process X percolates.

In the case where no admissible edge exists the condition of the above Lemma is satisfied. Now let e and x be as before. Consider the cluster of o in each iteration $(\mathcal{C}(\eta_s, o))_{s \leq t}$, and let $(C_s)_{s \leq t}$ be an admissible (deterministic) choice for these random clusters. We require

$$\mathbb{P}[C_t \leftrightarrow \Lambda_x \text{ in } \eta_t \cup \eta_x \mid \mathcal{C}(\eta_t, o) = C_t] \geq q.$$

We can not directly use the stopping set lemma here because the connection event we are considering contains the cluster we need to remove for the stopping set lemma. Instead we look to employ Lemma 6.3.

By assumption x is a neighbor of some cube that our process has already reached. This ensures that there is at least one box $x' + \Lambda_N$ which shares a quarter face with our target box Λ_x, such that $\text{diam}(C_t \cap \Lambda_{x'}) \geq n$. Furthermore, the box $x' + \Lambda_N \subset \Lambda_{3N}$

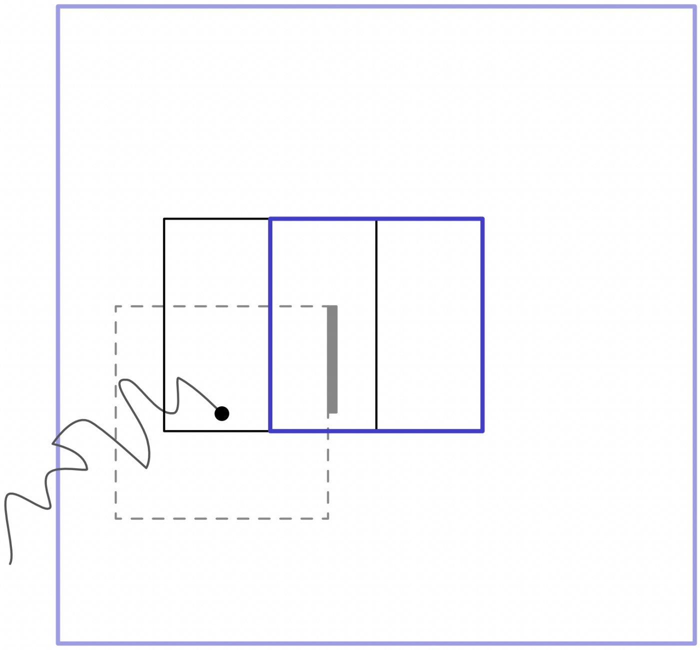

Figure 6-1: We can place a box around a point $x' \in C_t$ such that at least one quarter face of $x' + \Lambda_N$ is contained in Λ_x. The largest box represents $\tilde{\Lambda}_x$.

Thus, by Lemma 6.3:

$$\mathbb{P}[\psi_*^{C_t}\eta \leftrightarrow \Lambda_x \text{ in } \xi[\eta_{t+1}]] \geq 1 - \beta \exp(-c/\varepsilon).$$

The above display fulfills the condition of Lemma 6.5 to get

$$\mathbb{P}[\psi_*^{C_t}\eta_x \leftrightarrow \Lambda_x \text{ in } (\eta_t)_{\langle C_t \rangle} \cup \eta_x] \geq 1 - \delta.$$

By an application of the Stopping Set lemma we get

$$\begin{aligned} 1 - \delta &\leq \mathbb{P}[\psi_*^{C_t}\eta_x \leftrightarrow \Lambda_x \text{ in } (\eta_t)_{\langle C_t \rangle} \cup \eta_x] \\ &= \mathbb{P}[C_t \leftrightarrow \Lambda_x \text{ through } \eta_x \text{ in } \eta_t \cup \eta_x \mid \mathcal{C}(\eta_t, o) = C_t] \\ &= \mathbb{P}[C_t \leftrightarrow \Lambda_x \text{ in } \eta_t \cup \eta_x \mid \mathcal{C}(\eta_t, o) = C_t]. \end{aligned}$$

Now by Lemma 6.6 we are finished. ∎

6.2 Improving Uniqueness

We can now apply the Grimmett-Marstrand result to get better bounds on the uniqueness event $U(r, R)$. The approach is to first establish a better bound on connection within a box, and use that to ensure connection within an annulus via an "orange peeling" argument from [Gri99].

For all $t, L \in \mathbb{R}_{\geq 1}$ we define $T_t(L)$ to be $[-t,t]^{d-1} \times [-L, L]$. The following lemma and proof are adapted from [Gri99, Lemma 7.78]

Lemma 6.7. *Let $d \geq 3$ and $\lambda > \lambda_c$. There exist $L \in \mathbb{R}_{\geq 1}$ and a strictly positive constant $\delta = \delta(\lambda, L)$ such that*

$$\mathbb{P}[x \leftrightarrow y \textit{ in } \xi^{x,y} \cap T_t(L)] \geq \delta,$$

for all $x, y \in T_{t-L}(\frac{L}{3\sqrt{d}})$ and for all $t \geq L$.

Proof. Let $q = \frac{1}{3\sqrt{d}}$. We define the slab $S_t(L) := [-t,t]^2 \times [-L, L]^{d-2}$. We first prove that there exists some δ_0 and $L > 1$ sufficiently large such that

$$\mathbb{P}[x \leftrightarrow y \text{ in } \xi^{x,y} \cap S_t(L)] \geq \delta_0, \tag{6.5}$$

holds for all $x, y \in S_{t-L}(qL)$.

Let $Q^{NE}(L) = [-L, \infty)^2 \times [-L, L]^{d-2}$, where 'NE' stands for northeast. We similarly define Q^{NW}, Q^{SW}, Q^{SE}. Since $\lambda > \lambda_c$, we may assume by Theorem 6.1 that we can find some L such that $\lambda > \lambda_c(Q^{NE}(qL))$.[1] Let us write

$$\theta := \mathbb{P}[o \xleftrightarrow{Q^{NE}(qL)} \infty] > 0.$$

Let $s, l > 0$. We define $G^{NE}(s,l) := Q^{NE}(l) \cap \Lambda_{s+l}$ to be the intersection of the NE quadrant with a box. We define G^{NW}, G^{SW} and G^{SE} in an analogous manner. Let

$$H_{\uparrow}^{NE}(s,l) := [-l, s+l] \times [s+l, s+l+1] \times [-l, l]^{d-2},$$
$$H_{\rightarrow}^{NE}(s,l) := [s+l, s+l+1] \times [-l, s+l] \times [-l, l]^{d-2}$$

be the upper and right exterior boundary respectively. We define $H_{\uparrow}^{NW}$, $H_{\leftarrow}^{NW}$, $H_{\leftarrow}^{SW}$, $H_{\downarrow}^{SW}$, $H_{\downarrow}^{SE}$ and $H_{\rightarrow}^{SE}$ accordingly.

[1] By inspection of the proof and the fact that $p_c^{\text{site}}(\mathbb{Z}^2_{\geq 0}) > 0$.

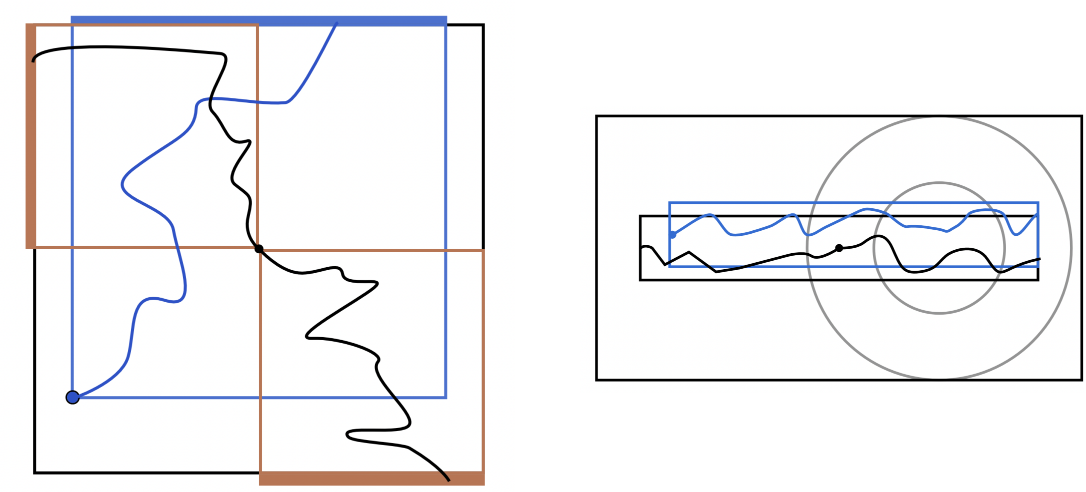

Figure 6-2: The left diagram shows the connection event in (6.7) from the top. The right diagram shows the same event from the side. The point y is shown in blue, and the origin o is shown in black.

Since $\{o \xleftrightarrow{Q^{NE}(qL)} \infty\}$ implies $\{\cup_{i\in\{\uparrow,\rightarrow\}} o \xleftrightarrow{G^{NE}(s,L/2)} H_i^{NE}(s,qL)\}$ we find by the union bound and symmetry that

$$\mathbb{P}[o \xleftrightarrow{G^{NE}(s,qL)} H_{\uparrow}^{NE}(s,qL)] \geq \theta/2. \tag{6.6}$$

Without loss of generality let $y \in S_{t-L}(qL)$ such that $y_1 \leq y_2 \leq 0$. Consider the event that

$$\begin{aligned} E_y = &\{o \xleftrightarrow{G^{NW}(t-L,qL)} H_{\leftarrow}^{NW}(t-L,qL)\} \\ &\cap \{o \xleftrightarrow{G^{SE}(t-L,qL)} H_{\downarrow}^{SE}(t-L,qL)\} \\ &\cap \{y \xleftrightarrow{y+G^{NE}(t-L+|y_1|,qL)} H_{\uparrow}^{NE}(t-L+|y_1|,qL)\}. \end{aligned} \tag{6.7}$$

In words, the above event ensures that the origin connects to the west side of the northwest quadrant and the south side of the of southeast quadrant of $S_{t-L}(qL)$. Furthermore, it ensures that y connects to the north side of the box. This event is illustrated in Figure 6-2.

By geometry, the event E_y ensures that the paths of y and o 'cross' somewhere in the two-dimensional sense in the box $[-t+L, t-L]^2$. By the nature of the MRCM the closest vertices will not directly overlap, but they will be within a distance of 1 (again in the two-dimensional sense). In the remaining $d-2$ dimensions the maximum distance is dictated by the thickness of the slab and is thus $3qL$ in every dimension. Thus, by the Pythagorean theorem, the maximum distance that the cluster of y and the cluster of o could be separated in the event (6.7) is at most L.

There is a scenario where the paths do not have to cross, if y is sufficiently close to the left side of the box that the left boundary of $y + G^{NE}(t - L + |y_1|, qL)$ is outside the inner slab $S_{t-L}(qL)$. But this path can at most protrude by qL, and so the above argument holds. Note that although we assumed $y_1 \leq y_2 \leq 0$, we can find an event E_y for all $y \in S_{t-L}(qL)$ by making the needed adjustments.

The paths crossing is not sufficient for o and y to connect. We can ensure connectivity using a sprinkling argument. Suppose the event E_y holds for $\lambda' = \frac{\lambda_c+\lambda}{2}$, which can be achieved by fixing a sufficiently large L. Then, there is a strictly positive, albeit small, probability that the closest points of the clusters $\mathcal{C}_o$ and $\mathcal{C}_y$ connect via only the sprinkled points of intensity $\lambda - \lambda'$. More precisely, for L fixed as above, the infimum $\inf_{x,y \in B_{L/2}} \mathbb{P}[x \overset{B_L}{\longleftrightarrow} y \text{ in } \xi[\eta^{x,y}_{\lambda-\lambda'}]] =: \delta_1$ is strictly positive.[2] Furthermore, as the connection is only along sprinkled points, it is independent of E_y.

Hence, by independence of the sprinkling event, the FKG inequality, and (6.6) we find

$$\mathbb{P}[y \leftrightarrow o \in \xi^{y,o} \cap S_t(L)] \geq \delta_1(\frac{1}{2}\theta)^3 =: \delta_2 > 0.$$

Finally, to recover (6.5) we use the FKG inequality again, together with our bound on the *Arm* event from Corollary 5.7. Then,

$$\begin{aligned}\mathbb{P}[x \leftrightarrow y \text{ in } \xi^{x,y} \cap S_t(L)] &\geq \mathbb{P}[E_x, E_y, Arm_o(L)^c] \\ &\geq \delta_2^2 - \mathbb{P}[Arm(L)],\end{aligned}$$

where we use the union bound and then the FKG inequality to get the final line. Hence, by Corollary 5.7, and choosing L sufficiently large, we find that (6.5) holds.

[2]This can be demonstrated by explicitly constructing a path. See e.g. [FPR11] for a similar construction.

Let $x = (x_1, \dots, x_d) \in T_{t-L}(qL)$. We construct the following sequence of points

$$\begin{aligned}
s_0 &= x \\
s_1 &= (0, x_2, x_3, \dots, x_d) \\
s_2 &= (0, 0, x_3, \dots, x_d) \\
&\vdots \\
s_{d-3} &= (0, \dots, 0, x_{d-2}, x_{d-1}, x_d) \\
s_{d-2} &= (0, \dots, 0).
\end{aligned}$$

We claim that

$$\mathbb{P}[s_j \leftrightarrow s_{j+1} \text{ in } \xi^{s_j, s_{j+1}} \cap T_t(L)] \geq \delta_0 \qquad \text{for } j \in [\![0, d-3]\!]\,. \tag{6.8}$$

First note that

$$\begin{aligned}
s_j, s_{j+1} \in\ & (0, \dots, 0, x_{j+3}, \dots, x_{d_2}, x_{d-1}, 0) \\
& + [-qL, qL]^j \times [-t + L, t - L]^2 \times [-qL, qL]^{d-j-2}.
\end{aligned}$$

The region on the right is a rotated and shifted version of $S_{t-L}(qL)$ that is a subset of $T_t(L)$. We can similarly place the larger slab $S_t(L)$ around the same point in the above display, and it is also a subset of $T_t(L)$. Hence, (6.8) follows from our earlier claim (6.5).

Hence, by FKG and Corollary 5.7

$$\mathbb{P}[x \leftrightarrow y \text{ in } \xi^{x,y} \cap T_t(L)] \geq \delta_0^{2(d-2)} - (2d-3)\mathbb{P}[Arm(L)].$$

Hence, we can find some L sufficiently large that the Lemma holds. ∎

The following lemma and proof are adapted from [Gri99, Lemma 7.89]

Lemma 6.8. *Let $\lambda > \lambda_c$. Then there exists some $\alpha, r_0 \in \mathbb{R}_{>1}$ and $c \in \mathbb{R}_{>0}$ such that for all $r \geq r_0$ it holds that*

$$\mathbb{P}[U(r, \alpha r)] \geq 1 - \exp(-cr).$$

Proof. The core of this proof relies on an "orange peeling" argument from [Gri99]. We work on a modified version of the uniqueness with d-cubes instead of balls to better apply Theorem 6.1. We can do this without problem by inscribing and circumscribing a ball around the relevant cubes, which will yield the same result up to some constant factors.

We start in a similar manner to Lemma 5.9. We consider $x, y \in \partial^{\text{in}} B_r$. We define the event

$$U(r, \alpha r; x, y) := \{x \leftrightarrow \Lambda_{\alpha r}^c, y \leftrightarrow \Lambda_{\alpha r}^c, x \overset{\Lambda_{\alpha r}}{\not\leftrightarrow} y\}.$$

Let q be as in Lemma 6.7. We pick L and $\delta = \delta(\lambda, L)$ like in Lemma 6.7 and L large enough such that

$$\delta^{d+2} - (d+1)\mathbb{P}[Arm((1-q)L)] \geq c, \tag{6.9}$$

for some $c > 0$.

Then we can write for all $r \geq 3L$

$$\alpha r = r + (2+q)LK$$

for some $K > 0$.

Let $r' := r + (1+q)L$. For $k \in [\![0, \lfloor K \rfloor]\!]$ and $x, y \in \partial^{\text{in}} \Lambda_r$ we define

$$A_k(x,y) := \{x \leftrightarrow \Lambda_{r'+k(2+q)L}^c, y \leftrightarrow \Lambda_{r'+k(2+q)L}^c, x \overset{\Lambda_{r+k(2+q)L}}{\not\leftrightarrow} y\},$$

to be the event that x and y leave the box $\Lambda_{r'+k(2+q)L}$, without connecting in the smaller box $\Lambda_{r+k(2+q)L}$. These boxes are chosen so that we may use Lemma 6.7. The intuition is that $A_k(x,y)$ is the event that ensures that the components of x and y reach the next layer without connecting in the current layer. Notice that

$$U(r, \alpha r; x, y)^c \subseteq A_{\lfloor K \rfloor - 1}(x,y) \subseteq A_{\lfloor K \rfloor - 2}(x,y) \subseteq \cdots \subseteq A_1(x,y),$$

and so,

$$\mathbb{P}[A_{\lfloor K \rfloor - 1}(x,y)] \leq \prod_{k=1}^{\lfloor K \rfloor - 1} \mathbb{P}[A_k(x,y) \mid A_{k-1}(x,y)].$$

We claim that

$$\mathbb{P}[A_k(x,y) \mid A_{k-1}(x,y)] \leq 1 - \delta',$$

for some $\delta' > 0$. This implies

$$\mathbb{P}[A_K(x,y)] \leq (1-\delta')^{\lfloor K \rfloor}.$$

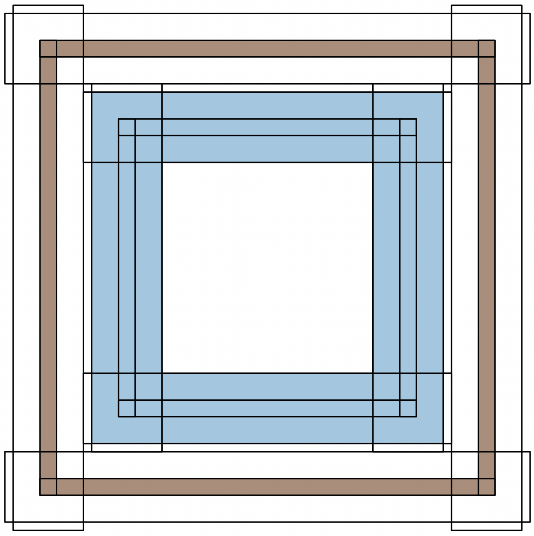

Figure 6-3: The blue inner ring represents D_k and the thin brown outer ring represents D'_k. The first 'peel' of the orange. Note the overshoot of the slabs is qL.

We define

$$D_k := \Lambda_{r+k(2+q)L-qL} \setminus \Lambda_{r+(k-1)(2+q)L}$$
$$D'_k := \Lambda_{r'+k(2+q)L} \setminus \Lambda_{r'+k(2+q)L-qL},$$

such that $D'_{k-1} \subset D_k$. The event $A_k(x,y)$ ensures that x and y reach D'_k without connecting in D_k. With an eye towards using Lemma 6.7, D_k will play the role of $T_t(L)$, whereas D'_k will play the role of $T_{t-L}(qL)$.

Now for any $z \in \eta \cap \Lambda_r$ we define $V_k(z)$ to be the set of all $u \in D'_k$. Notice that on the event $A_{k+1}(x,y)$ the sets $V_k(x)$ and $V_k(y)$ are non-empty and disjoint.

In other words, given that the event $A_{k-1}(x,y)$ occurs, the event $A_k(x,y)$ can occur only if for all $u \in V_{k-1}(x)$ and $v \in V_{k-1}(y)$ we have that $u \overset{D_k}{\not\longleftrightarrow} v$. This is a necessary, but not sufficient condition (which is exactly what we require for the following upper bound).

We can see that

$$\mathbb{P}[A_k(x,y) \mid A_{k-1}(x,y)] \leq \sup\{\mathbb{P}[u \not\leftrightarrow v \text{ in } \xi^{x,y} \cap D_k] : u,v \in D'_{k-1}\}.$$

Now we use Lemma 6.7 to prove that $\mathbb{P}[u \overset{D_k}{\longleftrightarrow} v \text{ in } \xi^{u,v}] \geq \delta^{d+2}$ for all $u,v \in D'_{k-1}$. The region D_k can be thought of as the union of $2d$ overlapping slices, where each slice is a shifted and rotated version of $T_{r_k}(L) = [-r_k, r_k]^{d-1} \times [-L, L]$, where $r_k :=$

$r - qL + k(2+q)L$. We also define $\tilde{r}_k := r - L + k(2+q)L$, which corresponds to $T_{r_k - L}(qL)$.

Let $z_0, \dots, z_d$ be the following collection of points in Λ_{r_k}. For each $i \in [\![0,d]\!]$ the first i entries of z_i are given by $\tilde{r}_k$ and the final $d-i$ entries are given by $-\tilde{r}_k$. In particular $z_0 = (-\tilde{r}_k, \dots, -\tilde{r}_k)$ and $z_d = (\tilde{r}_k, \dots, \tilde{r}_k)$.

For each consecutive pair z_i, z_{i+1}, there exists a copy of $T_{r_k - L}(qL)$ in D_k containing both z_i and z_{i+1}. Hence,

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{i \in [\![1,d]\!]} \{z_{i-1} \leftrightarrow z_i\}, \bigcap_{i \in [\![0,d]\!]} Arm_{z_i}((1-q)L)^c\right] \\
&\geq \mathbb{P}\left[\bigcap_{i \in [\![1,d]\!]} \{z_{i-1} \leftrightarrow z_i\}\right] - (d+1)\mathbb{P}[Arm((1-q)L)] \\
&\geq \delta^d - (d+1)\mathbb{P}[Arm((1-q)L)],
\end{aligned}$$

where we first use the union bound and then FKG.

Hence, by the same procedure as in the above display and our assumptions on L in (6.9), we find that

$$\begin{aligned}
\mathbb{P}[u \leftrightarrow v \text{ in } \xi^{u,v} \cap D_k] &\geq \mathbb{P}\left[u \leftrightarrow z_0, v \leftrightarrow z_d, \bigcap_{i \in [\![1,d]\!]} \{z_{i-1} \leftrightarrow z_i\}, \bigcap_{i \in [\![0,d]\!]} Arm_{z_i}(qL)^c\right] \\
&\geq \delta^{d+2} - (d+1)\mathbb{P}[Arm((1-q)L)] \geq c.
\end{aligned}$$

Note that we require the Arm events to ensure that the paths actually connect as it would not be sufficient for u to connect to v only through one of the z_i. ■

6.3 Proof of Theorem 5.1

To prove Theorem 5.1 we will require a lower bound on the probability that a randomly placed point in Λ_s reaches a large ball B_K at the origin. We achieve this by chaining together the previously defined uniqueness events with guarantees that the relevant annuli are crossed.

6.3.1 Gluing paths

First we show that a uniformly randomly picked vertex in some box Λ_t reaches a large central ball with sufficiently high probability.

Lemma 6.9. *Let $\lambda > \lambda_c$. Let $\varepsilon > 0$. Then there exists a $K > 0$ and an $t_0 > 0$, such that for all $t > t_0$ we have*

$$\mathbb{P}_\lambda[V_t \overset{\Lambda_t}{\longleftrightarrow} B_K] \geq \theta(\lambda) - \varepsilon,$$

where V_t is a uniformly picked point in Λ_t with a mark assigned by ρ.

We first need the following definitions.

Definition 6.10 (k-dependence)**.** Let $(X_x)_{x\in\mathbb{Z}^d}$ be a Bernoulli[3] random field. For any $k \in \mathbb{N}$ we call X a k-dependent field if for all finite $A, B \subset \mathbb{Z}^d$ such that all vertices in A have a distance (say ℓ_∞) of at least k to all vertices in B, then the collections of random variables $(X_x)_{x\in A}$ and $(X_x)_{x\in B}$ are independent.

Definition 6.11 (Stochasic domination)**.** Let $(X_x)_{x\in\mathbb{Z}^d}$ and $(Y_x)_{x\in\mathbb{Z}^d}$ be two Bernoulli random fields. We say say that X stochastically dominates Y if for every increasing function $f : \{0,1\}^{\mathbb{Z}^d} \to [0,\infty)$ it holds that $\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)]$.

We will need the following Lemma as used in [Pen03, Theorem 9.12] and originally proved in [LSS97], see also [Gri99, Theorem 7.65].

Lemma 6.12. *Let $\varepsilon \in (0, 1/4)$, $k \in \mathbb{N}$ and $(X_z)_{z\in\mathbb{Z}^d}$ a k-dependent Bernoulli random field. Then there exists some $\varepsilon' \in (0, \varepsilon)$ such that if $\mathbb{P}[X_z = 1] \geq 1 - \varepsilon'$ for all $z \in \mathbb{Z}^d$, we find that (X_z) stochastically dominates independent Bernoulli site percolation with parameter $1 - \varepsilon$.*

Lemma 6.13. *Let $\varepsilon > 0$. Then there exists some $\varepsilon' > 0$ such that the following holds. Let $(X_z)_{z\in\mathbb{Z}^d}$ be a $(1 - \varepsilon')$-Bernoulli site percolation. Then we have for all $s \geq 1$ and all $v \in \Lambda_s \cap \mathbb{Z}^d$ that*

$$\mathbb{P}[o \overset{(X_z)\cap\Lambda_s}{\longleftrightarrow} v] \geq 1 - \varepsilon. \tag{6.10}$$

Proof of Lemma 6.13. This proof follows from a Peierls type argument. We first show that if $o \overset{\Lambda_s\cap\mathbb{Z}^d}{\not\longleftrightarrow} v$ then there must exist some 'blocking set' which prevents the connection. This step will be purely deterministic. We then estimate the size of this blocking set, which in turn will give us a bound on the probability that o connects to v.

Let $s > 0$ and $v \in \Lambda_s$. Let $\omega \in \{0,1\}^{\Lambda_s\cap\mathbb{Z}^d}$ be a configuration. Suppose $o \overset{\omega}{\not\longleftrightarrow} v$. Let us call $C = \mathcal{C}_o(\omega)$ the set of sites that are connected to the origin. Now let D be the connected component of v in $\Lambda_s \cap \mathbb{Z}^d \setminus C$ (i.e. ignoring the configuration of ω). Finally, we define $C^+ := D^c \supset C$.

[3] i.e. $\{0,1\}$-valued.

Now, C^+ and D are disjoint connected sets whose union makes $\Lambda_s \cap \mathbb{Z}^d$. Let $\Delta := D \cap \partial^{\text{ext}} C^+$ be all the sites in D which are adjacent to some site in C^+. By [Pen03, Lemma 9.6] we know that Δ is $*$-connected[4]. Furthermore, all sites of Δ are vacant in ω and every possible path in $\Lambda_s \cap \mathbb{Z}^d$ from o to v uses at least one vertex in Δ. We call any set fulfilling these three criteria a *blocking set*.

Therefore, the event $o \overset{\omega}{\not\leftrightarrow} v$ is equivalent to the existence of a blocking set. It is immediate that the cardinality $|\Delta|$ is greater than or equal to $\operatorname{diam}_\infty \Delta := \max\{\|x - y\|_\infty : x, y \in \Delta\}$. We claim that

$$\operatorname{diam}_\infty \Delta \geq \frac{d_\infty(o, \Delta) \wedge d_\infty(v, \Delta)}{4}, \tag{6.11}$$

where d_∞ is the distance in the ∞-norm.

Suppose the claim is false, then $4 \operatorname{diam}_\infty \Delta < d_\infty(\Delta, o)$. Now let $\square$ be the smallest d-cube which contains Δ. Naturally, $\operatorname{diam}_\infty \Delta = \operatorname{diam}_\infty \square$. Then by the triangle inequality we find that

$$\begin{aligned} d_\infty(o, \square) &\geq d_\infty(o, \Delta) - \operatorname{diam}_\infty \Delta \\ &\geq 3 \operatorname{diam}_\infty \Delta. \end{aligned} \tag{6.12}$$

By the same argument we also find that $d_\infty(v, \square) \geq 3 \operatorname{diam}_\infty \Delta$. This causes a contradiction against Δ being a blocking set as we can now construct a path from o to v that avoids $\square$ and thus Δ.

For this construction we can assume without loss of generality that all coordinates of $v = (v_1, \ldots, v_d)$ are positive. For each permutation π of $\{1, 2, \ldots, d\}$ we can define a path γ_π which starts at o and moves to v coordinate by coordinate according to the ordering of π. So for the first segment of the path γ_π moves along coordinate $\pi(1)$ from 0 to $z_{\pi(1)}$.

Let $Q_k := [\![-k \operatorname{diam}_\infty \Delta, k \operatorname{diam}_\infty \Delta]\!]^d$. Observe that by (6.12) Δ can not intersect Q_3 or $z + Q_3$. However, for any two distinct permutations π and π' it holds that

$$d_\infty(\gamma_\pi \setminus (Q_3 \cup z + Q_3), \gamma_{\pi'} \setminus (Q_3 \cup z + Q_3)) \geq 3 \operatorname{diam}_\infty \Delta.$$

Hence, Δ can block at most one of these paths, and so our claim (6.11) holds[5]. Further, note that all of these paths will be fully contained in Λ_s, and hence the claim holds for

[4] Any two points in Δ can be joined by a path in Δ of ℓ_∞-adjacent vertices.

[5] This is where we use that Δ is $*$-connected.

all s.

Finally, let $\varepsilon > 0$. Then

$$\begin{aligned}\mathbb{P}[o \nleftrightarrow z] &= \mathbb{P}[\exists \text{ a blocking set } \Delta] \\ &\leq \sum_{k=1}^{\infty} \mathbb{P}[\exists \text{ a blocking set } \Delta, |\Delta| = k].\end{aligned}$$

We can now use (6.11) to observe that the blocking set must have at least one vertex in Q_4 or $z + Q_4$, each of which has at most k^d vertices. We know by [Pen03, Lemma 9.3] that for any given starting point there are at most $(2^{3^d-1})^k$ $*$-connected sets of cardinality k. Finally, each vertex in Δ is open with probability ε'. Hence,

$$\mathbb{P}[o \nleftrightarrow z] \leq \sum_{k=1}^{\infty} \varepsilon'^k (2k^d)(2^{3^d-1})^k.$$

Hence, we can find an $\varepsilon' > 0$ small enough so that (6.10) holds. ■

In the following proof we will write $U_\lambda(s, t; z)$ for the event $U_\lambda(s, t)$ centered at $z \in \mathbb{R}^d$. We remind the reader that Assumption (**A2**) states that for every $K > 0$ it holds that $\mathbb{P}_\lambda[B_K \leftrightarrow \infty] \geq 1 - cK^{-\delta}$.

Proof of Lemma 6.9. Let $\varepsilon > 0$. We now want to define a dependent site percolation on $K\mathbb{Z}^d$ that is coupled to our MRCM in such a way that we may use Lemmas 6.12 and 6.13. We will say that a site $z \in K\mathbb{Z}^d$ is open if

$$\{B_K(z) \leftrightarrow B_{\alpha K}^c\} \cap U_\lambda(K, \alpha K; z)$$

holds. This guarantees that if we have a sequence of neighboring open sites in $K\mathbb{Z}^d$ that we can find a corresponding path in the MRCM. More explicitly, for any path $(z_i)_{i=0}^f \subset K\mathbb{Z}^d$ of open sites such that $\|z_i - z_{i+1}\|_1 \leq K$ for all $i \leq f - 1$, we can find a corresponding path from $B_K(z_0)$ to $B_K(z_f)$ in $\bigcup_{i=0}^f B_{\alpha K}(z_i)$ in ξ_λ. Note that by the union bound, Assumption (**A2**) and Lemma 6.8,

$$\mathbb{P}[z \text{ open}] \geq 1 - cK^{-\delta} - \exp(-cK),$$

which can be chosen arbitrarily close to 1. We can now define a Bernoulli random field $(X_x)_{x \in K\mathbb{Z}^d}$ on $K\mathbb{Z}^d$. In particular, by Lemmas 6.12 and 6.13 we can choose $K_1 \in \mathbb{R}_{\geq 0}$ sufficiently large so that $\mathbb{P}[v \xleftrightarrow{(X_z) \cap \Lambda_s} o] \geq 1 - \varepsilon/3$ holds uniformly over s and v.

Next we consider the random point V_t. One way for $\{V_t \overset{\Lambda_t}{\longleftrightarrow} B_K\}$ to occur, is to 'connect' V_t to the Bernoulli site percolation system on $K\mathbb{Z}^d$. Then we can use our connection result on the site percolation. Let us write $\pi_{K\mathbb{Z}^d}(V_t)$ to denote the nearest site in the lattice $K\mathbb{Z}^d$ to V_t. Let $R = (1 + \sqrt{d}/2)K$. The ball $B_R(V_t)$ will contain the ball $B_K(\pi_{K\mathbb{Z}^d}(V_t))$. Thus, the events $\{V_t \leftrightarrow \infty\}$ and $U_\lambda(R, \alpha R; V_t)$ will ensure that V_t connects to $B_K(\pi_{K\mathbb{Z}^d}(V_t))$. Fix $K_2 \in \mathbb{R}_{\geq 0}$ so that $\mathbb{P}[U_\lambda(R, \alpha R)] \geq 1 - \varepsilon/3$

Now we fix $K = \max\{K_1, K_2\}$. We still have to ensure that our path stays within our chosen box Λ_t. We fix $t_0 := \frac{\alpha R}{1-(1-\varepsilon/3)^{1/d}}$. Now, for every $t \geq t_0$ it holds that the probability that V_t is within distance αR of the boundary is less than $\varepsilon/3$. By the union bound we find that:

$$\begin{aligned}\mathbb{P}[V_t \overset{\Lambda_t}{\longleftrightarrow} B_K] &\geq \mathbb{P}[V_t \leftrightarrow \infty, U_\lambda(R, \alpha R; V_t), V_t \in \Lambda_{t-H(R)}, \pi_{K\mathbb{Z}^d}(V_t) \overset{\Lambda_t \cap \mathbb{Z}^d}{\longleftrightarrow} o,] \\ &\geq \theta(\lambda) - \varepsilon/3 - \varepsilon/3 - \varepsilon/3.\end{aligned}$$

Hence the result holds. ■

Remark 6.14. The weaker bound on U_λ given in Proposition 5.8 does not suffice to prove the above Lemma. The problem occurs as the outer ball grows too fast to utilize the renormalization argument. In particular, the dependence of the random Bernoulli field grows unbounded as $R \to \infty$.

6.3.2 Proof of Theorem 5.1

We essentially restate the proof given in [Pen22].

Proof of Theorem 5.1. Assume $\lambda > \lambda_c$. Let $\varepsilon > 0$ and choose $K > 0$ such that $\mathbb{P}[B_K \leftrightarrow \infty] > 1 - \varepsilon$, $\mathbb{P}[V_s \leftrightarrow B_K] > \theta(\lambda) - \varepsilon$ and $\mathbb{P}[U_\lambda(K, \alpha K)] > 1 - \varepsilon$, using Lemma 6.12 and Lemma 6.8 for the latter two. Consider the sum

$$N_s := \sum_{x \in \eta_\lambda \cap \Lambda_s} 1\{x \leftrightarrow B_K \text{ in } \xi_\lambda[\Lambda_s]\}.$$

The idea is that $L_1(\Lambda_s) \subset N_s$ with high probability. Let V_s and W_s be uniformly distributed points in Λ_s. By the Mecke formula we write $\mathbb{E}[N_s] = \lambda(2s)^d \mathbb{P}[V_s \overset{\Lambda_s}{\longleftrightarrow} B_K]$. Thus,

$$\liminf_{s \to \infty} s^{-d} \mathbb{E}[N_s] \geq \lambda(\theta - \varepsilon).$$

Next, let

$$N'_s := \sum_{x \in \eta_\lambda \cap \Lambda_s} 1\{|\mathcal{C}_x(\eta_\lambda \cap \Lambda_s)| \geq s^{1/2}\},$$

be the number of vertices in components of size at least $s^{1/2}$. Using the Mecke formula we find that $\mathbb{E}[N'_s] = \lambda(2s)^d \mathbb{P}[|\mathcal{C}_{V_s}(\eta^{V_s} \cap \Lambda_s)| \geq s^{1/2}]$. Thus,

$$\lim_{s\to\infty} (2s)^{-d}\mathbb{E}[N'_s] = \lambda\theta.$$

Using the Mecke formula for double sums we find that

$$E[N'_s(N'_s-1)] = \lambda^2(2s)^{2d}\mathbb{P}[|\mathcal{C}_{V_s}(\eta_\lambda^{V_s,W_s} \cap \Lambda_s)| \geq s^{1/2}, |\mathcal{C}_{W_s}(\eta_\lambda^{V_s,W_s} \cap \Lambda_s)| \geq s^{1/2}].$$

Notice that the two events in the above display are independent whenever V_s and W_s have a distance of more than $2s^{1/2}$. The probability that V_s and W_s are closer than $2s^{1/2}$ is of order $s^{-d/2}$. Hence,

$$\lim_{s\to\infty} (2s)^{-2d}\mathbb{E}[N'_s(N'_s-1)] = \lambda^2\theta^2,$$

which in particular means that $(2s)^{-d}N'_s \to \lambda\theta$ in L^2 and hence in probability.

We see by a simple coupling that $(N_s - N'_s)_+ \leq \eta(B_{K+s^{1/2}})$ must hold. It follows that $s^{-d}\mathbb{E}[(N_s - N'_s)_+] \to 0$ as $s \to \infty$. Hence,

$$\limsup_{s\to\infty} \mathbb{E}[s^{-d}(N'_s - N_s)_+] = \limsup_{s\to\infty} \mathbb{E}[s^{-d}(N'_s - N_s)] \leq \lambda\varepsilon$$

By Markov's inequality, $\limsup_{s\to\infty} \mathbb{P}[s^{-d}(N'_s - N_s) \geq \varepsilon^{1/2}] \leq \lambda\varepsilon^{1/2}$. It follows that

$$\begin{aligned}
&\limsup_{s\to\infty} \mathbb{P}[s^{-d}N_s \leq \lambda\theta - 2\varepsilon^{1/2}]\\
\leq &\limsup_{s\to\infty} \Big(\mathbb{P}[s^{-d}N'_s \leq \lambda\theta - \varepsilon^{1/2}] + \mathbb{P}[s^{-d}(N_s - N's) \leq -\varepsilon^{1/2}]\Big) \leq \lambda\varepsilon^{1/2}
\end{aligned}$$

By our choice of K it holds that $\mathbb{P}[U_\lambda(K, \alpha K)] \geq 1 - \varepsilon$. If s is sufficiently large and $U_\lambda(K, \alpha K)$ holds we find that $L_1 \geq N_s - \eta_\lambda(B_{\alpha K})$, since all points in N_s outside $B_{\alpha K}$ must lie in the same component. Therefore,

$$\begin{aligned}
&\limsup_{s\to\infty} \mathbb{P}[(2s)^{-d}L_1 \leq \lambda\theta - 3\varepsilon^{1/2}]\\
&\leq \limsup_{s\to\infty} (\mathbb{P}[(2s)^{-d}N_s \leq \lambda\theta - 2\varepsilon^{1/2}] + \mathbb{P}[(2s)^{-d}\eta(B_{\alpha K}) \geq \varepsilon^{1/2}] + \mathbb{P}[U_\lambda(K, \alpha K)^c])\\
&\leq \lambda\varepsilon^{1/2} + \varepsilon.
\end{aligned}$$

Conversely,

$$\mathbb{P}[(2s)^{-d}L_1 \geq \lambda(\theta + \varepsilon)] \leq \mathbb{P}[(2s)^{-d}N'_s \geq \lambda(\theta + \varepsilon)] \xrightarrow{s\to\infty} 0.$$

This with the previous display shows $s^{-d} L_1 \xrightarrow{\mathbb{P}} \lambda\theta$.

If $(2s)^d \lambda(\theta + \varepsilon) > s^{1/2}$ and $L_1 + L_2 \geq (2s)^d \lambda(\theta + \varepsilon)$ then either $N_s' \geq (2s)^d \lambda(\theta + \varepsilon)$ or $L_1 + s^{1/2} \geq (2s)^d \lambda(\theta + \varepsilon)$. Hence, $\mathbb{P}[(2s)^{-d}(L_1 + L_2) > \lambda(\theta + \varepsilon)] \to 0$. It follows that $(2s)^{-d} L_2 \xrightarrow{\mathbb{P}} 0$. ■

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